이학박사 학위논문

On the emergent behaviors of stochastic flocking models

(확률적 플로킹 모형에서의 창발 현상에 관하여)

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이 논문을 이학박사 학위논문으로 제출함

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On the emergent behaviors of stochastic flocking models

A dissertation
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by

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Abstract

We introduce the Inertial Spin (IS) model and study its emergent dynamics under various frameworks. We first provide a derivation of Hamiltonian description of the IS model as a three dimensional flocking model with spin, which is an internal variable generating the rotation of velocities. We review some flocking estimates on the IS model and provide several numerical experiments. Then, we formally derive the Justh-Krishnaprasad (J-K) model as a two-dimensional restriction of the IS model under the small inertia regime. For the J-K model, we also review the flocking estimate of the J-K model and present an improved estimate from the previous one. As a mathematical model to describe a flocking behavior in nature, it is natural to assume a randomness on their dynamics. Thus, to provide a better description, one needs to incorporate such uncertain factors to the model and analyze their behaviors on the dynamics and stability of the flocking state. We here provide two different kind of noises on the J-K model and study the corresponding stochastic differential equations to fulfill this. Namely, we considered the update rules of velocity heading angles by adding Gaussian white noise to the update itself or adding the white noise to their coupling strength, which we call the additive and multiplicative noise, respectively. For the additive noise J-K model, we provide a lower-bound estimate on the probability of sample paths of heading angles to be confined in a certain bound in finite time, and also obtain an upper bound of expected order parameter square. For the multiplicative noise J-K model, we show that the multiplicative noise allows the asymptotic alignment of velocity heading angles if the coupling strength is sufficiently large compared to the diffusion.

Key words: Flocking, Inertial Spin model, Justh-Krishnaprasad model, Random dynamical system, Stochastic differential equation
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Contents

Chapter 1

Introduction

Emergent behaviors of many-body systems are ubiquitous in our nature. For example, aggregation of bacteria, flocking of birds, swarming of fishes, collective behaviors of pedestrians, synchronous chorusing of circada and firing of fireflies, etc [1, 6, 7, 11, 12, 18, 21, 26, 35, 34, 32, 28, 61, 66, 78, 77, 79]. Among such collective movements, our main interest lies in the so-called *flocking* phenomenon, where self-driven particles adjust their velocities based on simple rules or limited environmental information so that they become organized into an ordered motion. Due to recent applications on the control of drones, driverless cars and sensor networks [59, 64, 71, 74, 75], such emergent behavior has been a hot topic in control theory community.

Recently, several Vicsek type particle models with unit speed constraint were proposed in literatures [2, 14, 19, 20, 21, 22] for the study of velocity alignment. Among them, the Inertial Spin (IS) model [2, 14] is one of the most succesful one, having unit speed constraint with the conservation of certain internal variable called spin. Physically, the spin corresponds to the angular momentum of velocity vector, and conserved in the absence of external forces to the system. More precisely, let x_i, v_i and s_i be the position, velocity and spin of the i-th particle with generalized moment of inertia $\chi > 0$ in $\mathbb{R}^3$, respectively. Then, the dynamics of the Inertial Spin model is governed by the following system of ODEs:

$$
\begin{aligned}
&\dot{x}_i = v_i, \quad t > 0, \quad i = 1, \cdots, N, \\
&\dot{v}_i = \frac{1}{\chi} s_i \times v_i, \\
&\dot{s}_i = v_i \times \Big[\frac{\kappa}{N} \sum_{j=1}^{N} p_{ij} v_j - \gamma \dot{v}_i \Big],
\end{aligned}
\tag{1.0.1}
$$

where the initial data $\{(x_{i0}, v_{i0}, s_{i0})\}_{i=1}^N$ satisfies the constraints below:

$$
(x_i, v_i, s_i)\Big|_{t=0+} = (x_{i0}, v_{i0}, s_{i0}), \quad s_{i0} \cdot v_{i0} = 0, \quad |v_{i0}| = 1, \quad i = 1, \cdots, N.
\tag{1.0.2}
$$

Here, p_{ij} denotes an interaction rate between i-th and j-th particles, and γ, κ represent universal scales of damping and coupling, respectively.

When the inertia χ is suffieciently small, the IS model can be approximated by the ODE system without an internal variables $\{s_i\}_{i=1}^N$, which still has a unit speed constraint. In fact, as a result of approximation, we obtain the following Cucker-Smale like model with unit speed constraint:

$$
\begin{aligned}
&\dot{x}_i = v_i, \quad t > 0, \quad i = 1, \cdots, N, \\
&\dot{v}_i = \frac{\kappa}{N} \sum_{j=1}^{N} p_{ij} (v_j - (v_i \cdot v_j) v_i), \\
&v_i(0) = v_{i0}, \quad |v_{i0}| = 1, \quad i = 1, \cdots, N.
\end{aligned}
\tag{1.0.3}
$$

After proposed in [19], the model (1.0.3) draws a lot of attentions because of the similarity with Cucker-Smale model, and there has been a lot of literatures studied on (1.0.3) and its variants. The emergence of mono-cluster flocking [19], bi-cluster flocking [21], multi-cluster flocking and critical coupling strength [52], interplay with time delay [20] are studied. In particular, if the system (1.0.3) is restricted to a certain plane, one can introduce an heading angle variable describing the direction of each velocity in polar coordinate. This model is first presented by Justh and Krishnaprasad for constant p_{ij} before (1.0.3) and studied its possible applications on controlling UAVs [57, 58]. Then, the emergence of flocking is also studied for the time-dependent p_{ij} in [45] and named as the generalized Justh-Krishnaprasad (J-K) model. More precisely, let x_i be the position of the i-th J-K particle in $\mathbb{R}^2$ with unit speed, and the angle θ_i represents the corresponding direction of the velocity. Then, the dynamics of J-K particles (x_i, θ_i) is governed by the following second order system:

$$\begin{aligned} \frac{dx_j}{dt} &= (\cos\theta_j, \sin\theta_j), \quad j = 1, \cdots, N, \quad t > 0, \\ \frac{d\theta_j}{dt} &= \frac{\kappa}{N} \sum_{k=1}^{N} \psi(\|x_k - x_j\|) \sin(\theta_k - \theta_j). \end{aligned} \tag{1.0.4}$$

Here, the function $\psi : \mathbb{R} \to \mathbb{R}_+$ giving the time-dependent coefficient $p_{ij} = \psi(\|x_i - x_j\|)$ is called a communication weight and assumed to be non-increasing and globally Lipschitz continuous:

$$\begin{aligned} &[\psi]_{\mathrm{Lip}} := \sup_{r_1 \neq r_2} \frac{|\psi(r_1) - \psi(r_2)|}{|r_1 - r_2|} < \infty, \\ &(\psi(r_1) - \psi(r_2))(r_1 - r_2) \leq 0, \quad \forall\ r_1, r_2 \geq 0. \end{aligned} \tag{1.0.5}$$

For this model, we study the emergent behavior of the two stochastic variants of J-K model. First, we consider an sample path analysis and evolution of some expectations for the J-K system with additive white noises.

Specifically, we consider the J-K system with additive white noise in the dynamics of heading angles:

$$\begin{aligned} dx_t^j &= (\cos\theta_t^j, \sin\theta_t^j)\ dt, \quad t > 0,\ j = 1, \cdots, N, \\ d\theta_t^j &= \frac{\kappa}{N} \sum_{k=1}^{N} \psi(\|x_t^k - x_t^j\|) \sin(\theta_t^k - \theta_t^j) dt + \sqrt{2\sigma} dB_t^j, \\ (x_0^j, \theta_0^j) &= (x_{in}^j, \theta_{in}^j) \in \mathbb{R}^2 \times \mathbb{R}, \quad j = 1, \cdots, N, \end{aligned} \tag{1.0.6}$$

where B_t^j are independent and identically distributed Brownian motions. Since the right-hand side of (1.0.6) is Lipschitz continuous and uniformly bounded in state variables, the standard existence theory for SDEs implies the global-in-time existence and uniqueness of the strong solution. On the other hand, due to the additive white noise terms, the continuous sample trajectory of (1.0.6) will depart from the velocity heading angle alignment state even if those heading angles are initially aligned. Thus, the alignment state is unstable in the sense that there always exists a continuous sample path depart from the small neighborhood of alignment state asymptotically. Still, if the coupling κ is sufficiently larger than the diffusion σ, then the attracting drift effect might dominate the dynamics of system (1.0.6) for each pair (i, j) with $|\theta_i - \theta_j| \sim \frac{\pi}{2}$, since the distribution of noise is uniform in the whole domain. Therefore, we are interested in the following stochastic persistency problems:

1. "For a given initial configuration close to alignment state and finite-time window $[0, T]$, what will be the probability that the stochastic flow stays in a certain neighborhood of the alignment state?"
2. "For the J-K model with additive noise, is there any functional acts as an indicator of the instability of flocking state?"

To fix the idea, observe that $D(\Theta_t) := \max_{i,j} |\theta_i - \theta_j| = 0$ denotes the alignment of heading angles. So for a given T and D^∞, the probability

$$\mathbb{P}\Big\{ \sup_{0 \le t \le T} D(\Theta_t) < D^\infty \Big\} \tag{1.0.7}$$

will measure how much the configuration deviates from the alignment state during the time window $[0, T]$. Since the stochastic J-K flow is nonlinear, we might not be able to have an exact probability (1.0.7) depending of T and D^∞. Therefore, our primary goal for (1.0.7) is to provide a non-trivial lower bound for such probability in terms of system parameters. Note that even if the attraction between θ_i and θ_j is maximalized when $|\theta_i - \theta_j|$ is close to $\frac{\pi}{2}$, the coupling strength between them can take an arbitrary small values if the communication weight $\psi(\|x_i - x_j\|)$ is sufficiently small. This means that if given two particles are sufficiently far from each other, the interaction between them is negligible and the noise term will deterine the dynamics of those particles. In this case, we only pay attention on each small local systems that are close to each other but relatively far from the outside of the system, so that the communication weights in the local system have a positive lower bound. For example, for the case of velocity alignment of Myxobacteria [35, 34, 50, 51], it is natural to consider a system in each small growth medium so that each particles can affect to any others with nontrivial communication weight. Also, the existence of positive lower bound of ψ is also make sense as long as we consider the emergent behavior in finite time.

On the other hand, to estimate the instability of flocking state, we consider another indicator functional of flocking state. First, observe that the heading angles are all same up to modulo 2π, if and only if

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\cos(\theta^i-\theta^j)=1.$$

Therefore, if we can find an upper bound of the above quantity which is less than 1, then it immediately implies the instability of flocking states. In fact, the above functional is exactly same as the square of order-parameter $R(\Theta)$ of angle configuration $\Theta=(\theta^1,\cdots,\theta^N)$, which act crucial roles in the study of weakly coupled oscillator systems [22, 24, 27, 28, 41, 47, 49, 61, 62].

Next, we consider the J-K system with additive noise in the system parameters, such as coupling strength κ and communication weight ψ. For example, if each interaction from k to i is affected by the white noise as

$$\kappa\tilde{\psi}(\|x_t^i-x_t^k\|)=\kappa\psi(\|x_t^i-x_t^k\|)+\sqrt{2\sigma}\dot{B}_t^i,$$

we have the following stochastic J-K model with a multiplicative noise:

$$\begin{aligned}
&dx_t^j=(\cos\theta_t^j,\sin\theta_t^j)\;dt,\quad t>0,\;\; j=1,\cdots,N,\\
&d\theta_t^j=\frac{\kappa}{N}\sum_{k=1}^{N}\psi(\|x_t^k-x_t^j\|)\sin(\theta_t^k-\theta_t^j)dt+\frac{\sqrt{2\sigma}}{N}\sum_{k=1}^{N}\sin(\theta_t^k-\theta_t^j)dB_t^j,\\
&(x_0^j,\theta_0^j)=(x_{in}^j,\theta_{in}^j)\in\mathbb{R}^2\times\mathbb{R},\quad j=1,\cdots,N,
\end{aligned}\tag{1.0.8}$$

where $B_t^1,\cdots,B_t^N$ are independent one-dimensional Browian motions. Then, the scale of diffusion depends on the configuration θ. In particular, at the heading angle alignment state

$$\theta^1\equiv\cdots\equiv\theta^N\quad\text{modulo }2\pi,$$

the noise term also vanished with the drift term, so that we may expect the asymptotic stability of heading angle alignment state unlike the previous case (1.0.6).

Similarly, if a noise is added into the universal coupling scale κ as

$$\tilde{\kappa} = \kappa + \sqrt{2\sigma}\dot{B}_t,$$

we then have another stochastic J-K model:

$$\begin{aligned}
&dx_t^j = (\cos\theta_t^j, \sin\theta_t^j)\ dt, \quad t > 0,\ j = 1, \cdots, N, \\
&d\theta_t^j = \left(\frac{1}{N} \sum_{k=1}^{N} \psi(\|x_t^k - x_t^j\|) \sin(\theta_t^k - \theta_t^j) \right) (\kappa dt + \sqrt{2\sigma} dB_t), \\
&(x_0^j, \theta_0^j) = (x_{in}^j, \theta_{in}^j) \in \mathbb{R}^2 \times \mathbb{R}, \quad j = 1, \cdots, N.
\end{aligned} \tag{1.0.9}$$

Here, the noise term is also affected by the communication weight ψ, and therefore become smaller if ψ is small. As a result, we can even expect an asymptotic flocking when ψ does not have a positive lower bound.

Our results in Chapter 4 provide some partial answers to the above questions. First, we consider (1.0.8) and (1.0.9) for two particle cases and analyze the difference between two heading angles to provide a sufficient framework to achieve the asymptotic heading angle alignment. On the other hand, for the many-body cases, we assume the constant communication $\psi \equiv 1$ so that (1.0.8) and (1.0.9) both reduces to the Kuramoto model with multiplicative noises. Then, these two models only differs by the correlations of the white noises, and we study the temporal evolution of the expectation of $\log R(\Theta)$ in both cases.

The rest of this thesis is organized as follows. In Chapter 2, we briefly review the derivation of the IS model (1.0.1) by the Hamiltoniam formalism. Several previous results on its asymptotic behavior are also provided in this chapter, and we also present some numerical results. In Chapter 3, we give a heuristic derivation of the (deterministic) J-K model from the IS model and review the previous flocking estimates on it. We also present the improved flocking estimate for the deterministic J-K model, stochastic persistency analysis and expectation estimate on the order paramter for the J-K model with additive noise (1.0.6). Chapter 4 deals with the sufficient framework for the asymptotic heading angle alignment for (1.0.8) and (1.0.9) for two-particle cases and Kuramoto cases. Finally, Chapter 5 is devoted to the brief summary of this thesis and discussions on the possible future works.

Chapter 2

The Inertial Spin model

We briefly discuss a physical derivation of the Inertial Spin model and review some basic properties associated with the system (1.0.1)–(1.0.2), such as conservation/dissipation laws. Then, we study on asymptotic behavior of the IS system and provide some numerical examples for a more thorough understanding of the dynamics of the IS model. We note that this chapter is based on the joint work [8, 46].

2.1 A brief review on the IS model

We present the derivation procedure of system (1.0.1) following the literature [2, 13] and review its conservation and dissipation laws.

2.1.1 Derivation of the IS model

The experimental studies in [2, 14] found that starling flocks show the following behavior during their flight:

> *'Each bird in a flock move at (approximately) constant speed, during turning and straight flight. The speed is indeed uniform in whole flock and varies with species of birds.'*

To describe this experimental result, the authors in [2, 13] introduced a generalized coordinate that 'naturally' satisfies the above observation and derived their equations of motions. The detailed derivation of (1.0.1) is as below.

First, recall the Hamiltonian formalism for circular motion on xy plane, which is a motion of an object that moves its circular orbit. For the circular motion, we parametrize the orbit with its radius r and angular position θ, which is given as

$$\vec{\mathbf{r}} = (r\cos\theta, r\sin\theta, 0).$$

For this parametrization, we consider θ as a generalized coordinate of a point mass m and assume that the speed of m remains constant in the absense of external force(potential energy). In this case, the Lagrangian is given by the kinetic energy

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}mr^2\dot{\theta}^2$$

and the generalized momentum, canonical conjugate of θ is a derivative of $\mathcal{L}$ with respect to $\dot{\theta}$:

$$\text{Generalized momentum} = \frac{d\mathcal{L}}{d\dot{\theta}} = mr^2\dot{\theta} =: L.$$

The Hamiltonian for this generalized conjugate pair (θ, L) is the Legendre transform of Lagrangian $\mathcal{L}$:

$$H(\theta, L) = \dot{\theta}\cdot L - \mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}mr^2\dot{\theta}^2 = \frac{L^2}{2mr^2},$$

and we call the system parameter mr^2 as a moment of inertia I. For the rotation on generic two-dimensional subspace Π of $\mathbb{R}^3$, we define the angular momentum $\mathbf{L} = (L_1, L_2, L_3)$ as

$$\mathbf{L} := (r_1, r_2, r_3) \times (m\dot{r}_1, m\dot{r}_2, m\dot{r}_3), \quad r_1^2 + r_2^2 + r_3^2 = r^2 = \text{const.}$$

Then, we consider the three-dimensional angular position $(\theta_1, \theta_2, \theta_3)$ that satisfies

$$(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3) = \frac{1}{mr^2}\mathbf{L} = \frac{\mathbf{L}}{I},$$

so that the Lagrangian $\mathcal{L}$ and Hamiltonian H can be represented as

$$\mathcal{L}(\Theta, \dot{\Theta}) = \frac{1}{2}m(\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2) = \frac{|\mathbf{L}|^2}{2I}, \quad H\left(\Theta, \frac{d\mathcal{L}}{d\dot{\Theta}}\right) = \dot{\Theta} \cdot \frac{d\mathcal{L}}{d\dot{\Theta}} - \mathcal{L} = \mathcal{L},$$

and $\mathbf{L}$ itself is the canonical conjugate of Θ.

Similarly, consider a single agent moving on a certain plane $\Pi(\subset \mathbb{R}^3)$ at a constant speed v_0. Since the velocity $\mathbf{v}$ of the agent now moves on the plane Π, we imitate the above method and introduce a natural generalized coordinate for $\mathbf{v}$ which 'enforces' this constant speed constraint:

$$\mathbf{v} = \mathbf{p}\cos\varphi + \mathbf{q}\sin\varphi, \quad \mathbf{p}\cdot\mathbf{q} = 0, \quad |\mathbf{p}| = |\mathbf{q}| = v_0.$$

More precisely, we denote s as a generalized momentum canonically conjugate to the parameter φ, so that the Hamiltonian for this two-dimensional motion is now given as:

$$H(\varphi, s) := \frac{s^2}{2\chi},$$

where χ denotes a generalized moment of inertia. This H gives the equations of motions

$$\begin{aligned} \frac{d\varphi}{dt} &= \{\varphi, H\} = \frac{\partial H}{\partial s} = \frac{s}{\chi}, \\ \frac{ds}{dt} &= \{s, H\} = -\frac{\partial H}{\partial \varphi} = 0, \end{aligned}$$

and the corresponding velocity dynamics

$$\begin{aligned} \frac{d\mathbf{v}}{dt} = \{\mathbf{v}, H\} &= \frac{\partial \mathbf{v}}{\partial \varphi}\frac{\partial H}{\partial s} - \frac{\partial \mathbf{v}}{\partial s}\frac{\partial H}{\partial \varphi} \\ &= \frac{s}{\chi}(-\mathbf{p}\sin\varphi + \mathbf{q}\cos\varphi) \\ &= \frac{s}{\chi}\,(\hat{\mathbf{p}} \times \hat{\mathbf{q}}) \times (\mathbf{p}\cos\varphi + \mathbf{q}\sin\varphi) \\ &=: \frac{1}{\chi}\mathbf{s} \times \mathbf{v}. \end{aligned} \tag{2.1.1}$$

Then, we introduce a generalized three dimensional coordinate

$$\varphi := (\varphi^x, \varphi^y, \varphi^z),$$

and their corresponding spin $\mathbf{s} := (s^x, s^y, s^z)$, where the phases $(\varphi^x, \varphi^y, \varphi^z)$ satisfy

$$\frac{d\varphi}{dt} = \frac{\mathbf{s}}{\chi}.$$

We now return to the many-body system with alignment interactions. First, we define the Hamiltonian $H = H(V, S)$ for $V = (v_1, \cdots, v_N) \in \mathbb{R}^{3N}$ and $S = (s_1, \cdots, s_N) \in \mathbb{R}^{3N}$ as

$$H = H(V, S) := -\frac{\kappa}{2v_0^2} \sum_{i,j=1}^{N} n_{ij} v_i \cdot v_j + \frac{1}{2\chi} \sum_{i=1}^{N} |s_i|^2, \tag{2.1.2}$$

where (n_{ij}) and κ are symmetric connectivity matrix and the strength of alignment interaction, respectively. Although V and S are not conjugate variables, this (V, S) representation of Hamiltonian makes sense since the inner product of velocities are nothing but directional information if all velocities have fixed lengths. Then, for this Hamiltonian H, the equations of motions now become

$$\begin{aligned}
\frac{dx_i}{dt} &= v_i, \quad i = 1, \cdots, N, \\
\frac{dv_i}{dt} &= \frac{1}{\chi} s_i \times v_i, \\
\frac{ds_i}{dt} &= -v_i \times \frac{\partial H}{\partial v_i} = v_i \times \left(\frac{\kappa}{v_0^2} \sum_{j=1}^{N} n_{ij} v_j \right),
\end{aligned} \tag{2.1.3}$$

together with the constant speed condition and spin-velocity orthogonality condition, i.e.,

$$|v_i(t)| = v_0, \quad s_i(t) \cdot v_i(t) = 0, \quad t > 0, \quad i = 1, \cdots, N. \tag{2.1.4}$$

Moreover, the total spin $\mathbf{S} := \sum_j s_j$ is conserved along the flow:

$$\frac{d\mathbf{S}}{dt} = \frac{d}{dt} \left(\sum_{i=1}^{N} s_i \right) = \frac{\kappa}{v_0^2} \left(\sum_{i,j=1}^{N} n_{ij} v_i \times v_j \right) = 0.$$

We then consider a dissipation of spin to the system by adding

$$-v_i \times \left(\frac{\eta}{v_0^2} \frac{dv_i}{dt} \right) = -\frac{\eta}{\chi} s_i$$

in the right-hand side of $(2.1.3)_3$ and obtain

$$\frac{ds_i}{dt} = v_i \times \left(\frac{\kappa}{v_0^2} \sum_{j=1}^{N} n_{ij} v_j - \frac{\eta}{v_0^2} \frac{dv_i}{dt} \right), \tag{2.1.5}$$

where η denotes a viscous coefficient for spin. Finally, we derive (1.0.1) from (2.1.5) by slight modification of notations.

Remark 2.1.1. *In [2, 13], the authors considered the IS system in the presence of thermal bath so that the i.i.d vectorial noise* $\frac{v_i}{v_0} \times \xi_i$ *is added to the evolution of spin, where the variance* $\langle \xi_i \rangle$ *is proportional to the viscosity* η *and the (generalized) temperature* T. *Still, the condition* (2.1.4) *holds even in this stochastic IS model.*

2.1.2 Conservation laws

We now consider the dynamical variables of N particles (X, V, S), and assume that the dynamics of (X, V, S) is described by the (deterministic) IS system (1.0.1)–(1.0.2). Then, one can recover the conservation of speed and orthogonality of each (s, v) pair along the IS flow.

Proposition 2.1.1. *Let* (X, V, S) *be a solution to system* (1.0.1)–(1.0.2). *Then, we have the following conservation and dissipation laws.*

1. *The speed of each particle is a conserved quantity:*

$$\frac{d|v_i(t)|^2}{dt} = 0, \quad t > 0, \; i = 1, \cdots, N.$$

2. *The inner product of a spin and velocity is a conserved quantity:*

$$\frac{d}{dt}(s_i(t) \cdot v_i(t)) = 0, \quad t > 0, \; i = 1, \cdots, N.$$

3. The average spin decays to zero exponentially fast:

$$s_c(t) = s_c(0)e^{-\frac{\gamma t}{\chi}}, \quad s_c := \frac{1}{N}\sum_{i=1}^{N} s_i, \quad t > 0.$$

Proof. Note that (1) and (2) are immediate consequence of derivation procedure (2.1.1)–(2.1.5) for $\gamma = 0$. The purpose of this proposition is to verify if the IS model still has same property for nonzero damping γ.

• (Proof of 1): We take the inner product of both sides of $(1.0.1)_2$ with $2v_i$ to get

$$\frac{d}{dt}|v_i|^2 = \frac{2}{\chi} v_i \cdot (s_i \times v_i) = 0.$$

• (Proof of 2): Again, we use (1.0.1) to see that the t-derivative of $s_i \cdot v_i$ is identially zero for all i:

$$\begin{aligned}\frac{d}{dt}(s_i \cdot v_i) &= \dot{s}_i \cdot v_i + s_i \cdot \dot{v}_i \\ &= \left(v_i \times \left[\frac{\kappa}{N}\sum_{j=1}^{N} p_{ij} v_j - \gamma \dot{v}_i \right] \right) \cdot v_i + \frac{1}{\chi} s_i \cdot (s_i \times v_i) \\ &= 0.\end{aligned}$$

• (Proof of 3): We use $v_i \times (s_i \times v_i) = s_i$ to represent $(1.0.1)_3$ as

$$\dot{s}_i = v_i \times \left[\frac{\kappa}{N} \sum_{j=1}^{N} p_{ij} v_j \right] - \frac{\gamma}{\chi} s_i, \tag{2.1.6}$$

and then sum up (2.1.6) with respect to the index i to obtain

$$\dot{s}_c = \frac{1}{N}\sum_{i=1}^{N} \dot{s}_i = -\frac{\gamma}{\chi N}\sum_{i=1}^{N} s_i + \frac{\kappa}{N^2}\sum_{i,j=1}^{N} p_{ij} v_i \times v_j = -\frac{\gamma}{\chi N}\sum_{i=1}^{N} s_i = -\frac{\gamma}{\chi} s_c,$$

where we used $p_{ij} = p_{ji}$, $(v_i \times v_j) = -(v_j \times v_i)$ and index change $i \leftrightarrow j$. This yields the desired exponential decay. □

Remark 2.1.2. *1. The exponential convergence of total spin does not imply the exponential convergence of each s_i in general, even if each s_i converges to 0 asymptotically.*

2. *The IS system* (1.0.1) *conserves the quantities* $s_i \cdot v_i$ *and* $|v_i|$ *even if* $s_{i0} \cdot v_{i0}$ *and* v_{i0} *are not assumed to be* 0 *and* 1 *as in* (1.0.2). *In fact, an analogous result to Theorem 2.2.1 for the IS system with non-orthogonal* (s_i, v_i) *is provided in [8] (see Theorem 2.2.1 and Numerical examples in Section 2.3.1.2.)*

2.2 Asymptotic behavior of the IS model

We review the emergent behavior of the IS system in two different case, namely, the decoupled case and the coupled case. For the decoupled case, one can obtain that each particle moves a certain circular orbit, and each velocity will converge to a certain constant due to the nonzero damping γ.

2.2.1 Decoupled IS system

First, we present the explicit solution of the decoupled system, which can be obtained from (1.0.1) by turning off the social flocking force $\kappa = 0$. Then, the temporal evolution of each particle can be described by the following ODEs:

$$\begin{cases} \dot{x} = v, \quad \dot{v} = \dfrac{1}{\chi} s \times v, \quad \dot{s} = -\gamma v \times \dot{v}, \quad t > 0, \\ (x, v, s)|_{t=0+} = (x_0, v_0, s_0). \end{cases} \tag{2.2.1}$$

Proposition 2.2.1. *Let* (x, v, s) *be a solution to system* (2.2.1) *with initial data satisfying the relations:*

$$s_0 \neq 0, \quad s_0 \cdot v_0 = 0, \quad |v_0| = 1.$$

Then, the velocity $v(t)$ *has the following explicit form:*

$$v(t) = v_0 \cos\left[\frac{|s_0|}{\gamma}(1 - e^{-\frac{\gamma}{\chi}t})\right] + \frac{s_0 \times v_0}{|s_0|} \sin\left[\frac{|s_0|}{\gamma}(1 - e^{-\frac{\gamma}{\chi}t})\right].$$

Proof. First, regarding (x, v, s) as one-body IS system, Proposition 2.1.1 (3) implies that the spin $s(t)$ satisfies

$$s(t) = s_0 e^{-\frac{\gamma}{\chi}t}, \quad t > 0.$$

Then, since $v(t)$ has length 1 and orthogonal to $s(t)$ for all t, $v(t)$ can be represented as

$$v(t) = v_0 \cos\theta(t) + \frac{s_0 \times v_0}{|s_0|} \sin\theta(t), \quad t > 0. \tag{2.2.2}$$

Then, we substitute this ansatz (2.2.2) to $(2.2.1)_2$ to obtain

$$\begin{aligned} \dot{v}(t) &= \dot{\theta}(t) \left(-v_0 \sin\theta(t) + \frac{s_0 \times v_0}{|s_0|} \cos\theta(t) \right), \\ \frac{1}{\chi} s(t) \times v(t) &= e^{-\frac{\gamma}{\chi}t} \frac{|s_0|}{\chi} \left(-v_0 \sin\theta(t) + \frac{s_0 \times v_0}{|s_0|} \cos\theta(t) \right), \end{aligned}$$

so that we have

$$\dot{\theta} = e^{-\frac{\gamma}{\chi}t} \frac{|s_0|}{\chi}, \quad \theta(0) = 0. \tag{2.2.3}$$

Therefore, we deduce the desired result by integrating (2.2.3). □

2.2.2 Emergent dynamics of a many-body system

We now introduce an asymptotic alignment estimate for the many-body IS system with constant communication weights p_{ij}. For given static network topology (p_{ij}), the IS system (1.0.1)–(1.0.2) for observables $(X, V, S) = \{(x_i, v_i, s_i)\}_{i=1}^N$ can be written as:

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i = 1, \cdots, N, \\ \chi \dot{v}_i = s_i \times v_i, \\ \dot{s}_i = v_i \times \left[\dfrac{\kappa}{N} \displaystyle\sum_{j=1}^N p_{ij} v_j - \gamma \dot{v}_i \right], \\ (x_i, v_i, s_i)\Big|_{t=0+} = (x_{i0}, v_{i0}, s_{i0}), \quad s_{i0} \cdot v_{i0} = 0, \quad |v_{i0}| = 1. \end{cases} \tag{2.2.4}$$

Then, we can simplify the right-hand side of $(2.2.4)_3$ as in (2.1.6):

$$\begin{aligned} \dot{s}_i = v_i \times \left[\frac{\kappa}{N} \sum_{j=1}^N p_{ij} v_j - \gamma \dot{v}_i \right] &= \frac{\kappa}{N} \sum_{j=1}^N p_{ij} (v_i \times v_j) - \frac{\gamma}{\chi} v_i \times (s_i \times v_i) \\ &= \frac{\kappa}{N} \sum_{j=1}^N p_{ij} (v_i \times v_j) - \frac{\gamma}{\chi} s_i. \end{aligned} \tag{2.2.5}$$

Finally, since the subsystem of (2.2.4) consisting of $\{(v_i, s_i)\}_{i=1}^N$ is closed, we may ignore the dynamics of x and substitute (2.2.5) to $(2.2.4)_3$ to get the following system:

$$\begin{cases} \chi \dot{v}_i = s_i \times v_i, \quad i = 1, \cdots, N, \\ \dot{s}_i = \dfrac{\kappa}{N} \displaystyle\sum_{j=1}^N p_{ij}(v_i \times v_j) - \dfrac{\gamma}{\chi} s_i, \\ (v_i, s_i)\Big|_{t=0+} = (v_{i0}, s_{i0}), \quad s_{i0} \cdot v_{i0} = 0, \quad |v_{i0}| = 1. \end{cases} \tag{2.2.6}$$

Now, for a given solution (V, S) of (2.2.6), we set the following two functionals:

$$\mathcal{E} := \frac{1}{N^2} \sum_{i,j=1}^N p_{ij} |v_i - v_j|^2, \qquad \mathcal{S} := \frac{1}{N} \sum_{i=1}^N |s_i|^2.$$

Proposition 2.2.2. *Suppose that the parameters κ, γ and χ are all positive. If (V, S) is a solution to system (2.2.6), we have the following energy relation:*

$$\mathcal{E}(t) + \frac{2}{\chi\kappa} \mathcal{S}(t) + \frac{4\gamma}{\chi^2 \kappa} \int_0^t \mathcal{S}(\tau) d\tau = \mathcal{E}(0) + \frac{2}{\chi\kappa} \mathcal{S}(0), \quad t \geq 0.$$

Proof. First, we take the inner product of both sides of $(2.2.6)_2$ with $2s_i$ to get

$$\frac{d|s_i|^2}{dt} = \frac{2\kappa}{N} \sum_{j=1}^N p_{ij} s_i \cdot (v_i \times v_j) - \frac{2\gamma}{\chi} |s_i|^2. \tag{2.2.7}$$

Then, we sum the relation (2.2.7) over all i and use the index change argument $i \leftrightarrow j$ to obtain

$$\begin{aligned} \sum_{i,j=1}^N p_{ij} s_i \cdot (v_i \times v_j) &= \sum_{i,j=1}^N p_{ji} s_j \cdot (v_j \times v_i) = -\sum_{i,j=1}^N p_{ij} s_j \cdot (v_i \times v_j) \\ &= \frac{1}{2} \sum_{i,j=1}^N p_{ij} (s_i - s_j) \cdot (v_i \times v_j), \end{aligned}$$

so that the functional $\mathcal{S}$ satisfies

$$\frac{d}{dt} \mathcal{S} + \frac{2\gamma}{\chi} \mathcal{S} = \frac{\kappa}{N^2} \sum_{i,j} p_{ij} (s_i - s_j) \cdot (v_i \times v_j). \tag{2.2.8}$$

Therefore, it suffices to show that

$$\frac{d\mathcal{E}}{dt} = -\frac{2}{\chi N^2}\sum_{i,j} p_{ij}(s_i - s_j)\cdot(v_i \times v_j). \tag{2.2.9}$$

For each $i, j = 1, \cdots, N$, we have

$$\frac{d}{dt}(v_i - v_j) = \frac{1}{\chi}(s_i \times v_i - s_j \times v_j) = \frac{1}{\chi}\Big((s_i - s_j)\times v_i + s_j \times (v_i - v_j)\Big). \tag{2.2.10}$$

Then, we take the inner product of both sides of (2.2.10) with $2p_{ij}(v_i - v_j)$ to get

$$\begin{aligned}\frac{d}{dt}p_{ij}|v_i - v_j|^2 &= \frac{2}{\chi}p_{ij}(v_i - v_j)\cdot\Big((s_i - s_j)\times v_i + s_j \times (v_i - v_j)\Big)\\ &= \frac{2}{\chi}p_{ij}(v_i - v_j)\cdot(s_i - s_j)\times v_i = -\frac{2}{\chi}p_{ij}v_j\cdot((s_i - s_j)\times v_i)\\ &= \frac{2}{\chi}p_{ij}v_i\cdot((s_i - s_j)\times v_j) = -\frac{2}{\chi}p_{ij}(s_i - s_j)\cdot(v_i \times v_j).\end{aligned} \tag{2.2.11}$$

Finally, we average (2.2.11) over all i, j to derive (2.2.9). Then, $\frac{\chi}{2}$(2.2.9) + $\frac{1}{\kappa}$(2.2.8) yields

$$\frac{d}{dt}\left(\frac{\chi}{2}\mathcal{E} + \frac{1}{\kappa}\mathcal{S}\right) + \frac{2\gamma}{\chi\kappa}\mathcal{S} = 0, \quad t > 0,$$

which is the desired energy relation. □

Remark 2.2.1. *The above energy relation gives a conservation of* $\mathcal{E} + \frac{2}{\chi\kappa}\mathcal{S}$ *in the absence of damping* γ*. In fact, this functional is equivalent to the Hamiltonian in* (2.1.2) *up to constant.*

Proposition 2.2.2 implies that the functional

$$\mathcal{E}(t) + \frac{2}{\chi\kappa}\mathcal{S}(t)$$

is monotonically decreasing along the flow. However, its derivative is not strictly negative definite, and the set of equilibria of (2.2.6) forms a union of some manifolds. Namely,

$$\left\{(V,S) : \sum_{j=1}^{N} p_{ij}(v_i \times v_j) = 0, \quad s_i = 0, \quad \forall\, i = 1, \cdots, N\right\}.$$

Therefore, we cannot apply the Lyapunov's classical asymptotic stability argument. In such cases, one might use the LaSalle's invariance principle [63] or Barbalat's lemma [5] to show the convergence toward a positive invariant set without any decay rate. For the IS model, both can be applied to show the convergence of s_i's to zero, but we need to use the Barbalat's lemma for the convergence of their derivatives for higher orders. Therefore, we here introduce the Barbalat's lemma for later use (see Appendix A.1).

Lemma 2.2.1 ([5])**.** *Suppose that a real-valued function $f : [0,\infty) \to \mathbb{R}$ is uniformly continuous and satisfies*

$$\lim_{t\to\infty} \int_0^t f(s)ds \quad \textit{exists.}$$

Then, f tends to zero as $t \to \infty$:

$$\lim_{t\to\infty} f(t) = 0.$$

Now, we introduce the main result of this work, the velocity alignment estimate for the simplified IS system (2.2.6).

Theorem 2.2.1. *Suppose that the parameters κ, γ and χ are all positive. If (V,S) is a solution to system* (2.2.6)*, we have the following results:*

1. *All spins and their derivatives converge to* 0*:*

$$\lim_{t\to\infty} |s_i(t)| = 0, \quad \lim_{t\to\infty} \left|\frac{d^n s_i}{dt^n}\right| = 0, \quad i = 1, \cdots, N, \ n \geq 0.$$

2. *If the network topology (p_{ij}) has a multiplicative structure*

$$p_{ij} = p_i p_j, \quad p_i > 0, \quad i,j = 1, \cdots, N,$$

then the solution (V, S) converges to a stationary solution $(V^\infty, \mathbf{0})$, where the asymptotic velocity $V^\infty = (v_1^\infty, \cdots, v_N^\infty)$ satisfies

$$\text{either} \quad \sum_{i=1}^N p_i v_i^\infty = \mathbf{0} \quad \text{or} \quad v_i^\infty = \pm v_1^\infty \quad \forall\ i = 1, \cdots, N.$$

3. *If the network topology (p_{ij}) has a multiplicative structure and the initial configuration (V_0, S_0) satisfies*

$$\mathcal{S}(0) < \chi\kappa \left| \frac{1}{N} \sum_{i=1}^N p_i v_i(0) \right|^2, \tag{2.2.12}$$

we have

$$\sum_{i=1}^N p_i v_i^\infty \neq \mathbf{0}.$$

Moreover, when the condition

$$\mathcal{E}(0) + \frac{2}{\chi\kappa}\mathcal{S}(0) < \min_{1 \le j \le N} \left\{ \frac{8 p_j \sum_{i \neq j} p_i}{N^2} \right\} \tag{2.2.13}$$

holds, then the solution (V, S) exhibits an asymptotic flocking, i.e.,

$$v_1^\infty = \cdots = v_N^\infty.$$

Proof. • (Proof of 1): It follows from Proposition 2.2.2 that

$$\frac{4\gamma}{\chi^2\kappa} \int_0^t \mathcal{S}(\tau) d\tau \le \mathcal{E}(0) + \frac{2}{\chi\kappa}\mathcal{S}(0) < \infty, \quad t \ge 0. \tag{2.2.14}$$

Now, we verify that $\mathcal{S}$ is uniformly continuous, by showing $\dot{\mathcal{S}}$ is uniformly bounded. Indeed, it follows from (2.2.8) that

$$\begin{aligned} |\dot{\mathcal{S}}| &\le \frac{2\gamma}{\chi}\mathcal{S} + \frac{\kappa}{N^2}\sum_{i,j} p_{ij}|s_i - s_j| \cdot |v_i \times v_j| \le \frac{2\gamma}{\chi}\mathcal{S} + \kappa p_M \sqrt{2N\mathcal{S}} \\ &\le \frac{2\gamma}{\chi}\Big(\frac{\chi\kappa}{2}\mathcal{E}(0) + \mathcal{S}(0)\Big) + \kappa p_M \sqrt{N(\chi\kappa\mathcal{E}(0) + 2\mathcal{S}(0))} < \infty, \end{aligned} \tag{2.2.15}$$

where we used

$$p_M := \max_{1\le i\le N} p_{ij}, \quad |v_i| = 1, \quad \mathcal{S}(t) \le \mathcal{S}(t) + \frac{\chi\kappa}{2}\mathcal{E}(t) \le \mathcal{S}(0) + \frac{\chi\kappa}{2}\mathcal{E}(0),$$
$$|s_i - s_j|^2 \le (|s_i| + |s_j|)^2 \le 2(|s_i|^2 + |s_j|^2) \le 2N\mathcal{S}, \quad \text{i.e.,} \quad |s_i - s_j| \le \sqrt{2N\mathcal{S}}.$$

Thus, we use the Barbalat's lemma together with (2.2.14) and (2.2.15) to obtain

$$\lim_{t\to\infty} \mathcal{S}(t) = 0, \quad \text{i.e.,} \quad \lim_{t\to\infty} |s_i(t)| = 0, \quad i = 1, \cdots, N. \tag{2.2.16}$$

In addition, we can also show the convergence of derivatives $\dfrac{d^n s_i}{dt^n}$ to zero by using (2.2.8) and inductive argument as in (2.2.15).

• (Proof of 2): First, since s_i and $\dot{s}_i$ converge to zero, we apply these convergences to (2.2.6) to have

$$\lim_{t\to\infty} \left(\sum_{j=1}^N p_j v_j\right) \times v_i = 0, \quad \forall\ i = 1, \cdots, N.$$

On the other hand, observe that for multiplicative $p_{ij} = p_i p_j$, we have

$$\mathcal{E} = \frac{1}{N^2}\sum_{i,j=1}^N p_i p_j |v_i - v_j|^2 = 2\left(\frac{1}{N}\sum_{i=1}^N p_i\right)^2 - 2\left|\frac{1}{N}\sum_{i=1}^N p_i v_i\right|^2. \tag{2.2.17}$$

Since $\mathcal{S}$ converges to zero and $\mathcal{E} + \dfrac{2}{\chi\kappa}\mathcal{S}$ decrease monotonically, one can conclude that $\left|\dfrac{1}{N}\displaystyle\sum_{i=1}^N p_i v_i\right|$ converges to a nonnegative constant. Therefore, we split the problem into two cases.

⋄ Case A: First, assume that $w(t) := \dfrac{1}{N}\displaystyle\sum_{i=1}^N p_i v_i(t)$ converges to zero. Then by letting $t \to \infty$ in Proposition 2.2.2, we have

$$\mathcal{E}^\infty + \frac{4\gamma}{\chi^2\kappa}\int_0^\infty \mathcal{S}(u)du = \mathcal{E}(t) + \frac{2}{\chi\kappa}\mathcal{S}(t) + \frac{4\gamma}{\chi^2\kappa}\int_0^t \mathcal{S}(u)du, \tag{2.2.18}$$

which can be rewritten as

$$|w(t)|^2 + \frac{2\gamma}{\chi^2\kappa}\int_t^\infty \mathcal{S}(u)du = \frac{1}{\chi\kappa}\mathcal{S}(t) \tag{2.2.19}$$

as a consequence of (2.2.17) and $\mathcal{S} \to 0$. Now, we define a function g as an integration of $\mathcal{S}$ from t to infinity:

$$g(t) := \int_t^\infty \mathcal{S}(u)du \geq 0, \tag{2.2.20}$$

so that we have a Grönwall type inequalty for g:

$$g'(t) + \frac{2\gamma}{\chi}g(t) \leq 0, \quad t > 0.$$

This implies

$$0 \leq g(t) \leq g(0)e^{-\frac{2\gamma}{\chi}t}, \quad \forall t \in \mathbb{R}.$$

On the other hand, since the following relation holds:

$$\left(-g(t)e^{\frac{\gamma}{\chi}t}\right)' = \mathcal{S}(t)e^{\frac{\gamma}{\chi}t} - \frac{\gamma}{\chi}g(t)e^{\frac{\gamma}{\chi}t},$$

we obtain another improved estimate

$$\begin{aligned}\int_0^t \mathcal{S}(u)e^{\frac{\gamma}{\chi}u}du &= \int_0^t \frac{\gamma}{\chi}g(u)e^{\frac{\gamma}{\chi}u} + g(0) - g(t)e^{\frac{\gamma}{m}t} \leq \frac{\gamma}{\chi}g(0)\int_0^t e^{-\frac{\gamma}{\chi}u}du + g(0)\\ &\leq g(0)\left(1 - e^{-\frac{\gamma}{\chi}t}\right) + g(0) = 2g(0).\end{aligned}$$

Thus, we have

$$\int_0^\infty \mathcal{S}(u)e^{\frac{\gamma}{\chi}u}du < \infty, \quad \text{i.e.,} \quad \int_0^\infty |s_i(u)|^2 e^{\frac{\gamma}{\chi}u}du < \infty, \quad i = 1, \cdots, N.$$

From the relation $\chi|\dot{v}_i| = |s_i|$ in $(2.2.6)_1$, this is equivalent with

$$\int_0^\infty |\dot{v}_i(u)|^2 e^{\frac{\gamma}{\chi}u}du < \infty.$$

Therefore, we use the Hölder's inequality to see that $|\dot{v}_i|$ is in L^1:

$$\begin{aligned}\int_0^\infty |\dot{v}_i|du &= \int_0^\infty |\dot{v}_i(u)|e^{\frac{\gamma}{2\chi}u}e^{-\frac{\gamma}{2\chi}u}du\\ &\leq \left(\int_0^\infty |\dot{v}_i(u)|^2 e^{\frac{\gamma}{\chi}u}du\right)^{\frac{1}{2}}\left(\int_0^\infty e^{-\frac{\gamma}{\chi}u}du\right)^{\frac{1}{2}},\end{aligned}$$

and (V, S) converges to $(V^\infty, S^\infty) := (v_1^\infty, \cdots, v_N^\infty, 0, \cdots, 0)$ satisfying

$$\sum_i p_i v_i^\infty = \mathbf{0}.$$

$\diamond$ Case B: Now, assume that $|w(t)|$ converges to a positive constant. We here follow the proof in [8] especially for orthogonal (v, s) pairs. First, we claim:

$$\int_0^\infty |\dot{s}_i(t)|^2 dt < \infty. \tag{2.2.21}$$

To see this, consider the following relation on (V, S):

$$\begin{aligned}
\frac{d}{dt}(s_i \cdot \dot{s}_i) &= |\dot{s}_i|^2 + s_i \cdot \ddot{s}_i \\
&= |\dot{s}_i|^2 + s_i \cdot \frac{d}{dt}\left(\frac{\kappa}{N}\sum_{j=1}^N p_{ij}(v_i \times v_j) - \frac{\gamma}{\chi}s_i\right) \\
&= |\dot{s}_i|^2 - \frac{\kappa}{\chi N}\sum_{j=1}^N p_{ij}(s_i \times v_i) \cdot ((s_i - s_j) \times v_j) - \frac{\gamma}{\chi} s_i \cdot \dot{s}_i. \\
&= |\dot{s}_i|^2 - \frac{\kappa}{\chi N}\sum_{j=1}^N p_{ij}(s_i \times v_i) \cdot ((s_i - s_j) \times v_j) - \frac{\gamma}{2\chi}\frac{d|s_i|^2}{dt}.
\end{aligned} \tag{2.2.22}$$

Then, the integral $\int_0^\cdot$ of left-hand side and the last term in the right-hand side converges, since s_i and $\dot{s}_i$ converges to zero. Moreover, the second term in the right-hand side is integrable, since all s_i's are in L^2. Therefore, we first have the L^2-integrability of $\dot{s}_i$.

Now, since s_i and $\dot{s}_i$ converges to zero and in L^2, we apply these integrability results and $\lim_{t\to\infty}|w(t)| > 0$ to $(2.2.6)_2$ to deduce

$$v_i \times \frac{w}{|w|} \to 0 \quad \text{and} \quad v_i \times \frac{w}{|w|} \in L^2.$$

Therefore, for each i, we have

$$\text{either} \quad v_i - \frac{w}{|w|} \to 0 \quad \text{or} \quad v_i + \frac{w}{|w|} \to 0,$$

and since

$$1-\frac{|v_i\cdot w|}{|w|}\le 1-\frac{|v_i\cdot w|^2}{|w|^2}=\left|v_i\times\frac{w}{|w|}\right|^2,$$

we know that $v_i\pm\frac{w}{|w|}$ is contained in L^2 if it converges to zero.

It is now sufficient to prove the convergence of $\frac{w}{|w|}$. To see this, let I^+ be the set of indices i such that $v_i-\frac{w}{|w|}$ converges to zero, and I^- be the set of indices such that $v_i+\frac{w}{|w|}$ converges to zero, and define

$$u:=\frac{1}{N}\left(\sum_{i\in I^+}v_i-\sum_{i\in I^-}v_i\right).$$

Then, the time derivative of u can be written as

$$\dot{u}=\frac{1}{\chi N}\sum_{i\in I^+}s_i\times\left(v_i-\frac{w}{|w|}\right)-\frac{1}{\chi N}\sum_{i\in I^-}s_i\times\left(v_i+\frac{w}{|w|}\right)+\frac{1}{\chi}s_c\times\frac{w}{|w|},$$

so that the first and second terms in the right-hand side are in L^1. Moreover, since s_c converges to zero exponentially, we have $\dot{u}\in L^1$ and the convergence of u. Finally, since

$$u-\frac{w}{|w|}=\frac{1}{N}\sum_{i\in I^+}\left(v_i-\frac{w}{|w|}\right)-\frac{1}{N}\sum_{i\in I^-}\left(v_i+\frac{w}{|w|}\right)$$

converges to zero, we have the convergence of $\frac{w}{|w|}$ and $(v_1,\cdots,v_N)$.

• (Proof of 3) Note that our initial condition (2.2.12) allows a positive lower bound of $|w|$:

$$\begin{aligned}\mathcal{E}&=\frac{1}{N^2}\sum_{i,j=1}^N p_ip_j|v_i-v_j|^2=\frac{1}{N^2}\sum_{i,j=1}^N p_ip_j\Big(2-2(v_i\cdot v_j)\Big)\\&=2\left(\frac{1}{N}\sum_{j=1}^N p_j\right)^2-2|w|^2\le\mathcal{E}(0)+\frac{2}{\chi\kappa}\mathcal{S}(0)<2\left(\frac{1}{N}\sum_{j=1}^N p_j\right)^2,\end{aligned}$$

which excludes the asymptotic state $(v_1^\infty,\cdots,v_N^\infty)$ satisfying

$$\sum p_iv_i^\infty=\mathbf{0}.$$

Under (2.2.13), we calculate the limit of $|w|$ by following way:

$$\begin{aligned}
0 &= \lim_{t\to\infty} \left| w - \frac{1}{N}\sum_{i=1}^{N} p_i v_i \right| \\
&= \lim_{t\to\infty} \left| w - \frac{1}{N}\sum_{i\in I^+} p_i \frac{w}{|w|} + \frac{1}{N}\sum_{i\in I^-} p_i \frac{w}{|w|} \right| \\
&= \lim_{t\to\infty} \left| |w| - \frac{1}{N}\sum_{i\in I^+} p_i + \frac{1}{N}\sum_{i\in I^-} p_i \right| \\
&= \lim_{t\to\infty} \left| |w| - p_c + \frac{2}{N}\sum_{i\in I^-} p_i \right|.
\end{aligned} \tag{2.2.23}$$

However, the lower bound of $|w|$ can be computed as:

$$\begin{aligned}
|w|^2 &\geq p_c^2 - \frac{1}{2}\big(\mathcal{E}(0) + \frac{2}{\chi\kappa}\mathcal{S}(0)\big) \\
&> p_c^2 - \min_{1\leq i\leq N}\left\{\frac{4p_i(Np_c - p_i)}{N^2}\right\} = \max_{1\leq i\leq N}\left\{\frac{N^2p_c^2 - 4Np_ip_c + 4p_i^2}{N^2}\right\} \\
&= \max_{1\leq i\leq N}\left\{(p_c - \frac{2}{N}p_i)^2\right\},
\end{aligned}$$

where $p_c := \frac{1}{N}\sum_{i=1}^{N} p_i$. Combining these two results, we have

$$p_c - \frac{2}{N}\sum_{i\in I^-} p_i > \max_{1\leq i\leq N}\left|p_c - \frac{2}{N}p_i\right| \geq p_c - \frac{2}{N}p_j, \quad j = 1, \cdots, N,$$

and the positiveness of p_j implies $I^- = \emptyset$ and $I^+ = \{1, \cdots, N\}$. $\square$

Finally, we close this section with the linear stability analysis of the case $|w| \to 0$. For this, we introduce an auxiliary system for (2.2.6) by change of variable $w_i = p_i v_i$. For w_i, the system (2.2.6) can be rewritten as:

$$\begin{cases}
\chi\dot{w}_i = s_i \times w_i, \quad i = 1, \cdots, N, \\
\dot{s}_i = \dfrac{\kappa}{N}\displaystyle\sum_{j=1}^{N}(w_i \times w_j) - \dfrac{\gamma}{\chi}s_i, \quad i = 1, \cdots, N, \\
(w_i, s_i)\Big|_{t=0+} = (w_{i0}, s_{i0}), \quad s_{i0}\cdot w_{i0} = 0, \quad i = 1, \cdots, N.
\end{cases} \tag{2.2.24}$$

Theorem 2.2.2. *Let (V, S) be a solution to (2.2.6) together with the unit speed condition. If $\lim_{t\to\infty} |w(t)| = 0$, then (V, S) converges to some linearly unstable equilibrium.*

Proof. First, we write

$$\mathcal{I} = (w_1, \cdots, w_N, s_1, \cdots, s_N) \in \mathbb{R}^{6N},$$

and

$$w_i = (w_i^1, w_i^2, w_i^3) \in \mathbb{R}^3,\ \ s_i = (s_i^1, s_i^2, s_i^3) \in \mathbb{R}^3.$$

We construct $6N \times 6N$ Jacobian matrix $\mathbf{X}$ for $\mathcal{I}$

$$\mathbf{X} := \frac{\partial \dot{\mathcal{I}}}{\partial \mathcal{I}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are $3N \times 3N$ matrices as follows:

$$A = \frac{\partial \dot{w}_j}{\partial w_i}, \quad B = \frac{\partial \dot{w}_j}{\partial s_i}, \quad C = \frac{\partial \dot{s}_j}{\partial w_i}, \quad D = \frac{\partial \dot{s}_j}{\partial s_i}.$$

⋄ (Case of A): We use (2.2.4) and $s_i \to 0$ to find

$$\begin{aligned} A = \frac{\partial \dot{w}_j}{\partial w_i} &= \frac{\partial}{\partial w_i}\left(\frac{1}{\chi} s_j \times w_j\right) \\ &= \frac{1}{\chi}\frac{\partial}{\partial w_i}\Big(s_j^2 w_j^3 - s_j^3 w_j^2,\ s_j^3 w_j^1 - s_j^1 w_j^3,\ s_j^1 w_j^2 - s_j^2 w_j^1\Big) \\ &= \frac{1}{m}\begin{cases} 0 & (i \neq j) \\ \begin{pmatrix} 0 & -s_i^3 & s_i^2 \\ s_i^3 & 0 & -s_i^1 \\ -s_i^2 & s_i^1 & 0 \end{pmatrix} & (i = j) \end{cases} \\ &= 0. \end{aligned}$$

⋄ (Case of B): We also use (2.2.4) and $s_i \to 0$ to find

$$B = \frac{\partial \dot{w}_j}{\partial s_i} = \frac{1}{\chi}\begin{cases} 0 & (i \neq j) \\ \begin{pmatrix} 0 & w_i^3 & -w_i^2 \\ -w_i^3 & 0 & w_i^1 \\ w_i^2 & -w_i^1 & 0 \end{pmatrix} & (i = j). \end{cases}$$

$\diamond$ (Case of C): We use (2.2.5) to obtain

$$
\begin{aligned}
C &= \frac{\partial \dot{s}_j}{\partial w_i} = \frac{\partial}{\partial w_i}\left(\kappa(w_j \times w) - \frac{\gamma}{\chi}s_j\right) \\
&= \kappa\frac{\partial}{\partial w_i}\Big(w_j^2 w^3 - w_j^3 w^2,\ w_j^3 w^1 - w_j^1 w^3,\ w_j^1 w^2 - w_j^2 w^1\Big) \\
&= \begin{cases}
\frac{\kappa}{N}\begin{pmatrix} 0 & -w_j^3 & w_j^2 \\ w_j^3 & 0 & -w_j^1 \\ -w_j^2 & w_j^1 & 0 \end{pmatrix} & (i \neq j) \\
\frac{\kappa}{N}\begin{pmatrix} 0 & -w_j^3 & w_j^2 \\ w_j^3 & 0 & -w_j^1 \\ -w_j^2 & w_j^1 & 0 \end{pmatrix} + \kappa\begin{pmatrix} 0 & w^3 & -w^2 \\ -w^3 & 0 & w^1 \\ w^2 & -w^1 & 0 \end{pmatrix} & (i = j)
\end{cases} \\
&= \frac{\kappa}{N}\begin{pmatrix} 0 & -w_j^3 & w_j^2 \\ w_j^3 & 0 & -w_j^1 \\ -w_j^2 & w_j^1 & 0 \end{pmatrix}.
\end{aligned}
$$

$\diamond$ (Case of D): We also obtain

$$D = \frac{\partial \dot{s}_j}{\partial s_i} = \frac{\partial}{\partial s_i}\left(\kappa(w_j \times w) - \frac{\gamma}{\chi}s_j\right) = -\frac{\gamma}{\chi}\delta_{ij},$$

where δ_{ij} is the Kronecker delta.

For the handy notation, we write 3×3 matrix W_i as

$$W_i = \begin{pmatrix} 0 & w_i^3 & -w_i^2 \\ -w_i^3 & 0 & w_i^1 \\ w_i^2 & -w_i^1 & 0 \end{pmatrix}, \quad i = 1, \cdots, N.$$

Then, we can write the matrix B and C as

$$B = \frac{1}{\chi}\mathrm{diag}(W_1, \cdots, W_N) \quad \text{and} \quad C = -\frac{\kappa}{N}\begin{pmatrix} W_1 & \cdots & W_1 \\ \cdots & \ddots & \cdots \\ W_N & \cdots & W_N \end{pmatrix}.$$

Note that

$$\mathrm{tr}\left(-W_i^2\right) = 2(w_i^1)^2 + 2(w_i^2)^2 + 2(w_i^3)^2 = 2p_i^2. \tag{2.2.25}$$

Then, we compute the characteristic polynomial of $\mathbf{X}$ as follows: for each eigenvalue λ, we have

$$
\begin{aligned}
0 = \det\Big(\mathbf{X} - \lambda I_{6N}\Big) &= \det \begin{pmatrix} -\lambda I_{3N} & B \\ C & \left(-\lambda - \frac{\gamma}{\chi}\right) I_{3N} \end{pmatrix} \\
&= \det\left(\lambda\left(\lambda + \frac{\gamma}{\chi}\right) I_{3N} - BC\right) \\
&= \det\left(\lambda\left(\lambda + \frac{\gamma}{\chi}\right) I_{3N} - \frac{\kappa}{\chi N}\begin{pmatrix} -W_1^2 & \cdots & -W_1^2 \\ \cdots & \ddots & \cdots \\ -W_N^2 & \cdots & -W_N^2 \end{pmatrix}\right) \\
&=: \det\left(\lambda\left(\lambda + \frac{\gamma}{\chi}\right) I_{3N} - \mathcal{M}\right).
\end{aligned}
$$

It follows from (2.2.25) that the $3N \times 3N$ matrix $\mathcal{M}$ has a positive trace

$$
\mathrm{tr}\mathcal{M} = \frac{\kappa}{\chi N}\sum_{i=1}^{N} \mathrm{tr}\left(-W_i^2\right) = \frac{2\kappa}{N\chi}\sum_{i=1}^{N} p_i^2 > 0,
$$

and therefore $\mathcal{M}$ has an eigenvalue λ_0 whose real part is positive, i.e., $\mathrm{Re}\lambda_0 > 0$. For this λ_0, every $\lambda \in \mathbb{C}$ satisfying

$$
\lambda\left(\lambda + \frac{\gamma}{m}\right) = \lambda_0 =: c + \mathrm{i}d,
$$

is an eigenvalue of $\mathbf{X}$. Now if we write λ by $\lambda = a + \mathrm{i}b$, $(a, b, c, d) \in \mathbb{R}^4$, then the above relation implies

$$
a^2 - b^2 + \frac{\gamma}{\chi}a = \mathrm{Re}\lambda_0 = c, \quad 2ab + \frac{\gamma}{\chi}b = \mathrm{Im}\lambda_0 = d.
$$

Then,

$$
c = a^2 + \frac{\gamma}{\chi}a - \frac{d^2}{\left(2a + \frac{\gamma}{\chi}\right)^2} = a^2 + \frac{\gamma}{\chi}a - \frac{d^2}{4a^2 + \frac{4\gamma}{\chi}a + \frac{\gamma^2}{\chi^2}}
$$

and if we write $Z := a^2 + \dfrac{\gamma}{\chi}a$, we have

$$
4Z^2 + \left(\frac{\gamma^2}{\chi^2} - 4c\right)Z - d^2 - \frac{\gamma^2}{\chi^2}c = 0.
$$

Since $c = \mathrm{Re}\lambda_0 > 0$, the above quadratic equation attains two distinct real roots $z_1 < 0 < z_2$ and $Z = a^2 + \frac{\gamma}{\chi}a = b^2 + c$ has to be positive. Hence, we have

$$a^2 + \frac{\gamma}{\chi}a = z_2,$$

and the above equation always attains one positive real root. In other words, the matrix $\mathbf{X}$ has at least one eigenvalue whose real part is positive. Therefore, we can conclude that the equilibrium is linearly unstable. $\square$

2.3 Numerical simulations

We perform several numerical simulations for (1.0.1)–(1.0.2) and study qualitative dynamics on the dynamic variables such as velocity and spin by varying the parameters γ and κ. For all simulations, we used the fourth-order Runge-Kutta method under the following system parameters:

$$\chi = 1, \quad N = 100, \quad \Delta t = 0.01.$$

Then, we plotted 10 curves from those 100 curves, except Figure 2.7. In Figure 2.7, we plotted all 100 curves to show the non-flocking for fast exponential ψ.

2.3.1 Decoupled IS model

We now present numerical examples on the dynamics of *decoupled* inertial spin system with $\kappa = 0$:

$$\dot{x}_i = v_i, \quad \chi\dot{v}_i = s_i \times v_i, \quad \dot{s}_i = -\gamma v_i \times \dot{v}_i, \quad t > 0, \quad i = 1, \cdots, N.$$

2.3.1.1 Orthogonal (s_i, v_i)

Assume that

$$s_{i0} \cdot v_{i0} = 0, \quad i = 1, \cdots, 10.$$

Then, due to the rotational symmetry of (1.0.1) and decoupled structure, we may assume that

$$x_{i0} = (x_{i0}^1, x_{i0}^2, x_{i0}^3), \quad s_{i0} = (s_{i0}^1, 0, 0), \quad v_i = (0, v_{i0}^2, \sqrt{1 - |v_{i0}^2|^2}),$$

where the third condition follows from (1.0.2). The initial data set $(x_{i0}, s^1_{i0}, v^2_{i0})$ was randomly chosen from the five-dimensional cube $[-1, 1]^5$.

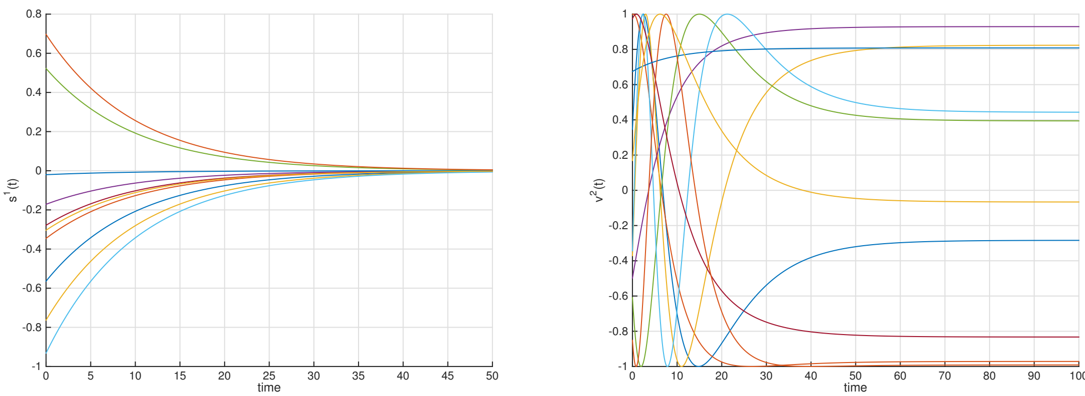
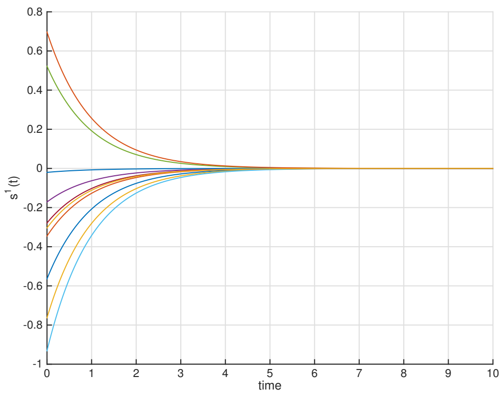


Figure 2.1: Evolution of s^1 and v^2 with $(\gamma, \chi) = (0.1, 1)$.

In Figure 2.1, we plot the first component of spins and the second component of velocities for (1.0.1) equipped with parameters $(\gamma, \chi) = (0.1, 1)$. As we can expect from the results in Section 2.2.1, s^1 converges to zero exponentially, and v^2 converges to some other values possibly nonzero. Since these 10 particles are not coupled at all, the limit of each velocities are obviously different.

In Figure 2.2, we also plot the first component of spins and the second component of velocities for (1.0.1)–(1.0.2) equipped with parameters $(\gamma, \chi) = (1, 1)$. Here, we can also observe the exponential decay of s^1 and convergence of v^2. However, as we increased the damping γ from 0.1 to 1 for fixed $\chi = 1$, exponential decay rate of s^1 becomes larger (see the time scales in Figure 2.1 and Figure 2.2).



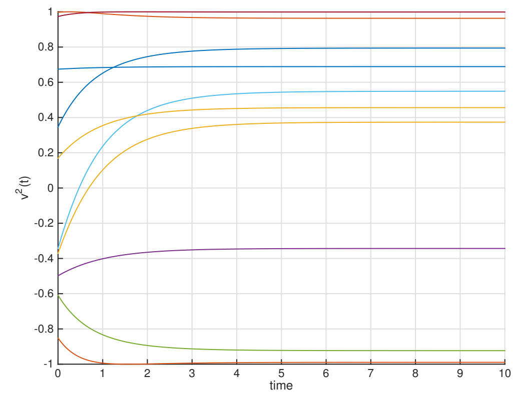


Figure 2.2: Evolution of s^1 and v^2 with $(\gamma, \chi) = (1, 1)$.

2.3.1.2 Non-orthogonal (s_i, v_i)

We provide the result of numeric simulations when $s_i \cdot v_i$ are initially nonzero. For simplicity, we assume

$$s_{i0} \cdot v_{i0} = 1, \quad |v_{i0}| = 1, \quad i = 1, \cdots, 10.$$

Then, due to the rotational symmetry of (1.0.1), we may assume that

$$x_{i0} = (x_{i0}^1, x_{i0}^2, x_{i0}^3), \quad s_{i0} = (s_{i0}^1, 0, 0), \quad v_{i0} = \left(\frac{1}{s_{i0}^1}, v_{i0}^2, \sqrt{1 - |v_{i0}^2|^2 - \left| \frac{1}{s_{i0}^1} \right|^2} \right).$$

Here, the initial data set (x_{i0}, v_{i0}^2) was randomly chosen from $[-1, 1]^4$ and s_{i0}^1 was randomly chosen to satisfy $|s_{i0}^1| \geq 1$.

In Figure 2.3, we plot s^2 and v^2 for system (1.0.1)–(1.0.2) with parameters $(\gamma, \chi) = (0.1, 1)$. Spin and velocity components s^2 and v^2 display oscillatory phase at the beginning, but they were saturated to some fixed values asymptotically. As noted in Remark 2.1.2, the convergence of these variables are proved in [8] even for coupled cases. Note that the limit of spins are also nonzero vectors in this case, which is different with Figure 2.1 and Figure 2.2.

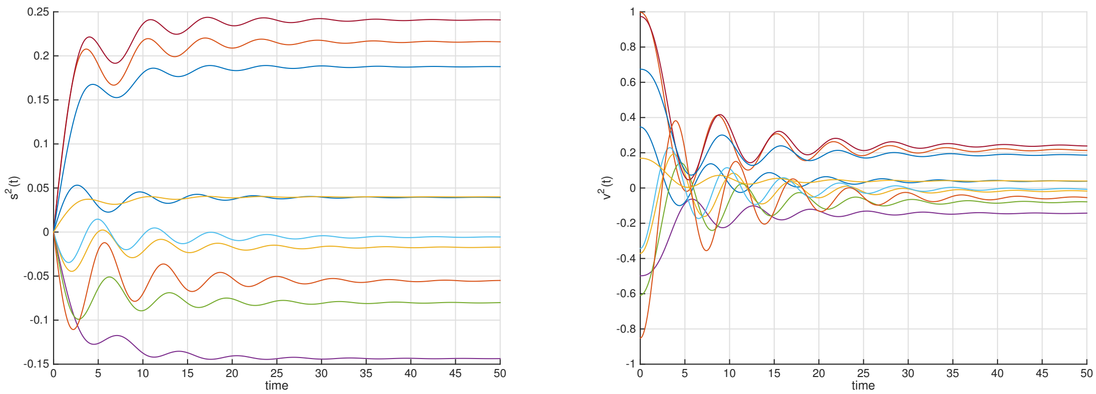


Figure 2.3: Evolution of s^2 and v^2: $(\gamma, \chi) = (0.1, 1)$.

2.3.2 A coupled IS model

Below, we perform several numerical simulations on the dynamics of the IS system with multiplicative communication weights:

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i = 1, \cdots, 100, \\ \chi \dot{v}_i = s_i \times v_i, \quad s_{i0} \cdot v_{i0} = 0, \quad |v_{i0}| = 1, \\ \dot{s}_i = v_i \times \left[\dfrac{\kappa}{100} \displaystyle\sum_{j=1}^{100} p_i p_j v_j - \gamma \dot{v}_i \right], \end{cases}$$

where each p_i is randomly chosen from the interval $(0, 1)$.

Recall that $s_i \cdot v_i$ and $|v_i|$ are conserved for all time. In the following simulations, we set

$$s_i \cdot v_i = 0, \quad |v_i| = 1.$$

Now, we figure out the role of coupling strength κ by comparing coupled and decoupled cases. To compare the results precisely, we choose the same initial configurations with Figure 2.1 and Figure 2.2.

In Figure 2.4, we plot the short-time and long-time evolutions of $s^1(t)$ with parameters

$$\gamma = 0.1, \quad \kappa = 0.3, \ 1, \ 10.$$

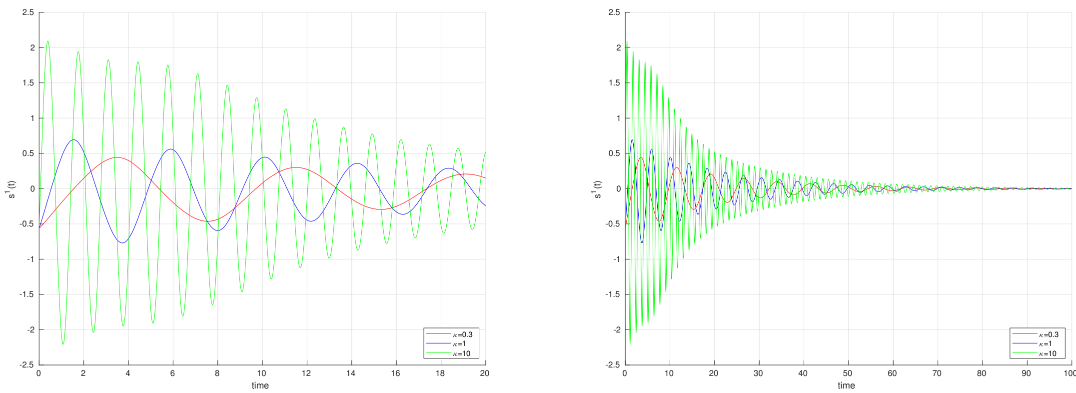


Figure 2.4: Long and short time evolutions of s^1 with $(\gamma, \chi) = (0.1, 1)$ and $\kappa = 0.3, 1, 10$.

As κ increases, the amplitude and frequency of oscillatory motions also increase. Despite its oscillatory behavior, its upper and lower envelopes converge to zero.

In Figure 2.5, we plot both short-time and long-time evolutions of v^2 with parameters:

$$\gamma = 0.1, \quad \kappa = 0.3,\ 1,\ 10.$$

However, the amplitude of oscillations of v^2 becomes smaller as coupling strength κ increases, while the frequency of the oscilliation increases as the spin does. Similar to the spin variable, v^2 converges to some common value asymptotically.

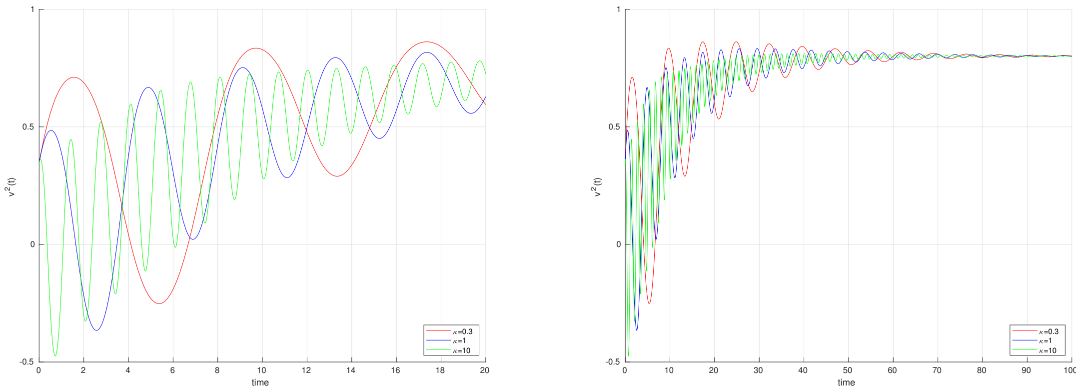


Figure 2.5: Long and short time evolutions of v^2 with $(\gamma, \chi) = (0.1, 1)$ and $\kappa = 0.3, 1, 10$.

In Figure 2.6, we plot s^1 and v^2 again with parameters

$$\gamma = 1, \quad \kappa = 0.3, \; 1, \; 10.$$

Although oscillatory motions of s^1 and v^2 appear during the evolution, it is saturated in short time due to the large friction.

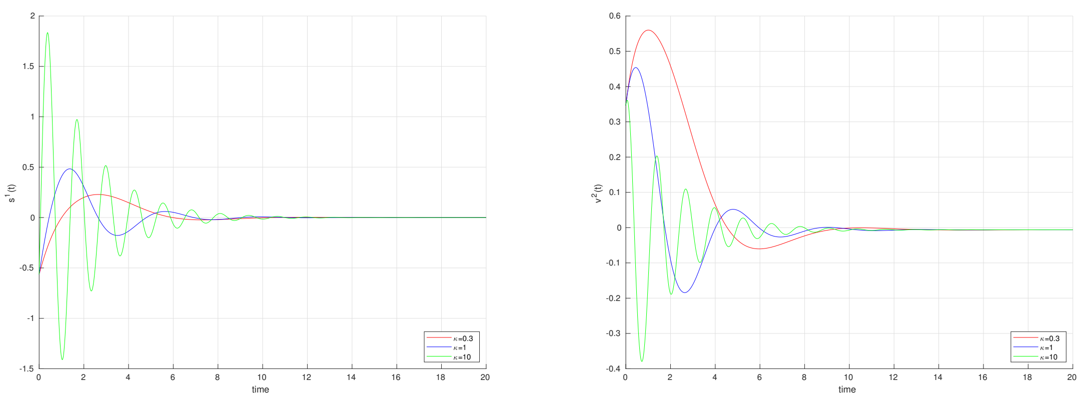


Figure 2.6: Evolution of s^1 and v^2: $(\gamma, \chi) = (1, 1)$ and $\kappa = 0.3, \; 1, \; 10$.

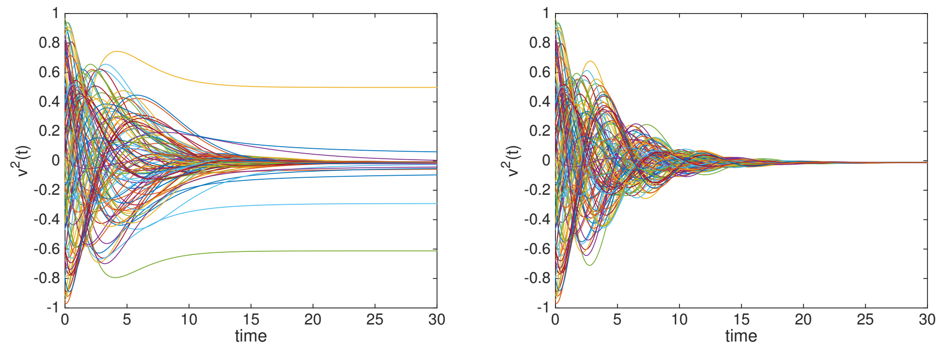


Figure 2.7: Flocking and non-flocking for different ψ

In Figure 2.7, we plot v_2 with parameters

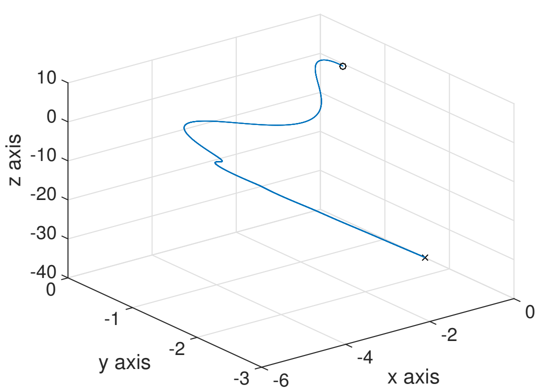
$$\gamma = 0.5, \quad \chi = 1, \quad \kappa = 1, \quad p_{ij} = \psi(\|x_i - x_j\|),$$

where $\psi(r) = e^{-r}$ for fast exponential and $\psi(r) = e^{-\frac{1}{2}r}$ for slow exponential. From the explicit solution formula of the decoupled IS model, the converging rates of spin and velocity are expected as $e^{-\frac{\gamma}{\chi}t} = e^{-\frac{1}{2}t}$ for constant p_{ij}. Thus, if the communication weight p_{ij} also decays exponentially, the asymptotic behavior of (1.0.1) might be affected. However, Figure 2.7 shows that even if p_{ij} decays exponentially, the flocking might occur if it is slow exponential.

In Figure 2.8, we plot 3D trajectories with parameters

$$\gamma = 0.5, \quad \chi = 1, \quad \kappa = 1,$$

and constant $p_{ij} = p_i p_j$ as in Figure 5,6,7. We choose the trajectory of 1 or 10 trajectories in the original model, which contains 100 particles. From this picture, one can verify that the presence of spins induce a collective turn of the system.



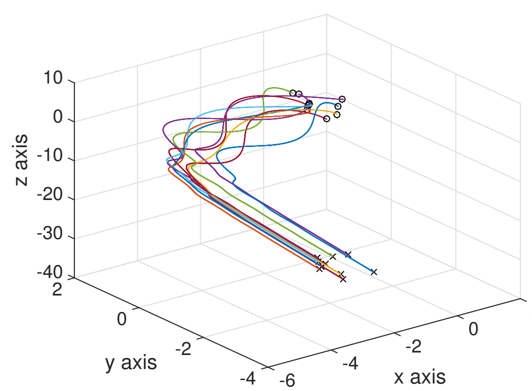


Figure 2.8: 3D trajectories

Chapter 3

Justh-Krishnaprasad model with additive noises

We study the emergent behavior of Justh-Krishnaprasad flocking model (1.0.4) and its stochastic counterpart (1.0.6) in the presence of additive noise. Recall that the stochastic J-K system is given as

$$
\begin{aligned}
& dx_t^j = (\cos \theta_t^j, \sin \theta_t^j)\ dt, \quad t > 0,\ j = 1, \cdots, N, \\
& d\theta_t^j = \frac{\kappa}{N} \sum_{k=1}^{N} \psi(\|x_t^k - x_t^j\|) \sin(\theta_t^k - \theta_t^j) dt + \sqrt{2\sigma} dB_t^j, \\
& (x_0^j, \theta_0^j) = (x_{in}^j, \theta_{in}^j) \in \mathbb{R}^2 \times \mathbb{R},
\end{aligned}
\tag{3.0.1}
$$

and the deterministic one can also be described by (3.0.1) as a special case $\sigma = 0$.

Throughout the chapter, we assume that the coupling strength κ is strictly positive and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive and uniformly Lipschitz nonincreasing function with Lipschitz constant $[\psi]_{\mathrm{Lip}}$. Then, the classical well-posedness theories of ODE and SDE ensure the global-in time existence and uniqueness of the solution $\{(x_t^j, \theta_t^j)\}_{j=1}^N$ of (1.0.4) and (3.0.1). Note that for the special case $\psi \equiv 1$ in the absence of noise ($\sigma = 0$), the dynamics of heading angles $(\theta_t^1, \cdots, \theta_t^N)$ can be reduced to the Kuramoto system, which has been studied in a lot of literature [24, 27, 55, 61].

In the sequel, we provide the detailed flocking estimates of (3.0.1), both deterministic case ($\sigma = 0$) and stochastic case ($\sigma > 0$). We first present a formal derivation of the deterministic J-K model (1.0.4) from the IS system introduced in Chapter 2, and then improve the flocking estiamte for the deterministic system (1.0.4) in [45]. For the system (3.0.1), however, we cannot expect the convergence of θ^j's as $t \to \infty$ due to the additive noise terms. Therefore, we provide a lower-bound estimate of the probability to stay close to the flocking state in a certain finite time interval, by using the probability estimate on the sample paths of Ornstein–Uhlenbeck (OU) process and comparison between sine function with its linearization near zero. Moreover, we also estimate the expectation of order parameter square R^2 for (3.0.1) for the case $\psi \equiv 1$ so that we can use the whole nonlinear dynamics of Kuramoto model with additive noise.

The rest of this chapter is organized as follows. In Section 3.1, we present a heuristic derivation of the J-K system from the IS system and review the basic properties of the deterministic and stochastic J-K systems. In Section 3.2, we present a new flocking estimate on the deterministic J-K system (1.0.4) improved from [45]. We then provide a detailed proof of probability estimate for the stability of flocking state and the behavior of expectation of order parameter square R^2 for $\psi \sim 1$ in Section 3.3 and Section 3.4, respectively. We note that this chapter is based on the joint work [50].

3.1 Preliminaries

We present the formal derivation procedure of the J-K model from the IS model and review the basic properties of the deterministic and stochastic J-K models.

3.1.1 From the IS model to the J-K model

First, we set the interaction weight p_{ij} by a quantity depending on the relative distance:

$$p_{ij} = \psi(\|x_i - x_j\|), \quad i, j = 1, \cdots, N.$$

Then, the equation of the IS system (1.0.1) can be written as

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i = 1, \cdots, N \\ \dot{v}_i = \dfrac{1}{\chi} s_i \times v_i, \\ \dot{s}_i = v_i \times \Big[\dfrac{\kappa}{N} \displaystyle\sum_{j=1}^{N} \psi(\|x_i - x_j\|) v_j - \gamma \dot{v}_i \Big], \end{cases} \tag{3.1.1}$$

subject to initial data $\{(x_{i0}, v_{i0}, s_{i0})\}_{i=1}^{N}$ satisfying

$$\|v_{i0}\| = 1, \quad s_{i0} \cdot v_{i0} = 0, \quad i = 1, \cdots, N.$$

We further differentiate $(3.1.1)_2$ with respect to t to obtain

$$\begin{aligned} \chi \ddot{v}_i &= \dot{s}_i \times v_i + s_i \times \dot{v}_i \\ &= \Big[v_i \times \Big(\frac{\kappa}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|) v_j - \gamma \dot{v}_i \Big) \Big] \times v_i + \frac{1}{\chi} s_i \times (s_i \times v_i) \\ &= \frac{\kappa}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|)(v_i \times v_j) \times v_i - \gamma (v_i \times \dot{v}_i) \times v_i + \frac{1}{\chi} s_i \times (s_i \times v_i) \\ &= \frac{\kappa}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|) \Big(v_j - (v_j \cdot v_i) v_i \Big) - \gamma \dot{v}_i - \frac{1}{\chi} |s_i|^2 v_i \\ &= \frac{\kappa}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|) \Big(v_j - (v_j \cdot v_i) v_i \Big) - \gamma \dot{v}_i - \chi |\dot{v}_i|^2 v_i, \end{aligned} \tag{3.1.2}$$

where we used the following identities:

$$\begin{aligned} &(v_i \times v_j) \times v_i = v_j - (v_j \cdot v_i) v_i, \quad (v_i \times \dot{v}_i) \times v_i = (v_i \cdot v_i) \dot{v}_i - (v_i \cdot \dot{v}_i) v_i = \dot{v}_i, \\ &s_i \times (s_i \times v_i) = (s_i \cdot v_i) s_i - (s_i \cdot s_i) v_i = -|s_i|^2 v_i, \\ &\chi^2 |\dot{v}_i|^2 = |s_i \times v_i|^2 = |s_i \times v_i|^2 + |s_i \cdot v_i|^2 = |v_i|^2 |s_i|^2 = |s_i|^2. \end{aligned}$$

Then, we take a formal zero inertia limit ($\chi \to 0$) in (3.1.2) and obtain an equation of velocity evolution:

$$\dot{x}_i = v_i, \quad \dot{v}_i = \frac{\bar{\kappa}}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|)\Big(v_j - (v_j \cdot v_i)v_i\Big), \quad \bar{\kappa} := \frac{\kappa}{\gamma}. \tag{3.1.3}$$

Now, if all initial positions and velocities $\{(x_{i0})\}_{i=1}^N, \{(v_{i0})\}_{i=1}^N$ are contained in a two-dimensional subspace Π, it is clear that the entire solution $\{(x_i, v_i)\}_{i=1}^N$ of (3.1.3) lies on the plane Π. Therefore, in this case, we can rewrite v_i as a polar coordinate form:

$$v_i = (\cos\theta_i, \sin\theta_i). \tag{3.1.4}$$

Then, we substitute the relation (3.1.4) into the equation (3.1.3) to obtain

$$\begin{aligned}&(-\sin\theta_i, \cos\theta_i)\dot{\theta}_i \\ &\quad = \frac{\kappa}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|)\Big[(\cos\theta_j, \sin\theta_j) - \cos(\theta_i - \theta_j)(\cos\theta_i, \sin\theta_i)\Big].\end{aligned}$$

Finally, we take the inner product of above equation with $(-\sin\theta_i, \cos\theta_i)$ to get the deterministic J-K model that have been studied in [45]:

$$\frac{dx_i}{dt} = (\cos\theta_i, \sin\theta_i), \quad \frac{d\theta_i}{dt} = \frac{\kappa}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|)\sin(\theta_j - \theta_i).$$

3.1.2 A brief review on the J-K model

Recall that the dynamics of deterministic J-K model (1.0.4) is governed by the following ODE system:

$$\begin{aligned}&\dot{x}_j = (\cos\theta_j, \sin\theta_j), \quad t > 0, \quad j = 1, \cdots, N, \\ &\dot{\theta}_j = \frac{\kappa}{N} \sum_{k=1}^{N} \psi(\|x_k - x_j\|)\sin(\theta_k - \theta_j).\end{aligned}$$

Then, not only the rotational symmetry inherited from the IS model, we can also see that the total sum of heading angles is conserved along the dynamics (1.0.4).

Proposition 3.1.1 ([45])**.** *Let $\{(x_i, \theta_i)\}_{i=1}^N$ be a smooth solution to system* (1.0.4) *with the initial data $\{(x_{i0}, \theta_{i0})\}_{i=1}^N$. Then, the total sum of heading angles $\{\theta_i\}_{i=1}^N$ is conserved along the dynamics* (1.0.4)*:*

$$\sum_{j=1}^N \theta_j(t) = \sum_{j=1}^N \theta_{j0}, \quad t \geq 0.$$

Proof. We sum $(1.0.4)_2$ over all j and use the index exchange trick $j \Longleftrightarrow k$:

$$\begin{aligned}\frac{d}{dt}\sum_{j=1}^N \theta_j &= \frac{\kappa}{N}\sum_{j,k=1}^N \psi(\|x_k - x_j\|)\sin(\theta_k - \theta_j) \\ &= -\frac{\kappa}{N}\sum_{j,k=1}^N \psi(\|x_j - x_k\|)\sin(\theta_j - \theta_k) = 0,\end{aligned}$$

to get the desired estimate. □

Now, we recall the definition of asymptotic flocking of the deterministic J-K system introduced in [45].

Definition 3.1.1. *For the deterministic J-K system* (1.0.4)*, we call the solution $\{(x_i, \theta_i)\}_{i=1}^N$ exhibits an asymptotic flocking if*

1. *All relative heading angles $(\theta_i - \theta_j)$ converge to zero modulo 2π as t goes to infinity (alignment of heading angles), i.e.,*

$$\lim_{t\to\infty} \cos(\theta_i(t) - \theta_j(t)) = 1, \quad \forall 1 \leq i, j \leq N.$$

2. *All relative distances of the configuration $\{x_i(t)\}_{i=1}^N$ is uniformly bounded in t:*

$$\sup_{0\leq t<\infty} \max_{i,j} \|x_i(t) - x_j(t)\| < \infty.$$

Instead of verifying the convergence $\cos(\theta_i - \theta_j) \to 1$ for each i, j, we represent the notion of heading angle alignment more efficient way by considering the order parameter. We here introduce the definition of order parameter R.

Definition 3.1.2. *For a given angle configuration $\Theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N$, the order parameter $R = R(\Theta)$ is defined as*

$$R(\Theta) := \left| \frac{1}{N} \sum_{k=1}^{N} e^{\mathrm{i}\theta_k} \right|. \tag{3.1.5}$$

Then, we can recharacterize the alignment of heading angles by using the order parameter of $\{\theta_j\}_{j=1}^N$.

Proposition 3.1.2 ([50])**.** *Let $\{(x_i, \theta_i)\}_{i=1}^N$ be the solution of equation* (1.0.4)*. Then, the configuration $\Theta = (\theta_1, \cdots, \theta_N)$ exhibits an alignment of heading angles if and only if*

$$\lim_{t\to\infty} R(\Theta(t)) = 1.$$

Proof. Since the order parameter $R(\Theta)$ is defined as (3.1.5), we have

$$\begin{aligned} R(\Theta)^2 &= \left(\frac{1}{N}\sum_{k=1}^N e^{\mathrm{i}\theta_k}\right) \cdot \overline{\left(\frac{1}{N}\sum_{k=1}^N e^{\mathrm{i}\theta_k}\right)} = \left(\frac{1}{N}\sum_{k=1}^N e^{\mathrm{i}\theta_k}\right) \cdot \left(\frac{1}{N}\sum_{j=1}^N e^{-\mathrm{i}\theta_j}\right) \\ &= \frac{1}{N^2}\sum_{j,k=1}^N e^{\mathrm{i}(\theta_k-\theta_j)} = \mathcal{R}e\left[\frac{1}{N^2}\sum_{j,k=1}^N e^{\mathrm{i}(\theta_k-\theta_j)}\right] = \frac{1}{N^2}\sum_{j,k=1}^N \cos(\theta_k-\theta_j). \end{aligned}$$

Therefore, the heading angle configuration Θ exhibits an alignment if and only if $R(\Theta)$ converges to 1 as $t \to \infty$. □

Although the order parameter R gives a simple description of the heading angle alignment, it is still hard to show the emergence of flocking from the estimation of order parameter for the J-K system (1.0.4). We now introduce the definition of two types of diameters of angle configuration.

Definition 3.1.3 ([50])**.** *For a given heading angle vector $\Theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N$, we define the diameters $D(\Theta)$ and $D_{\mathbb{T}}(\Theta)$ as*

$$D(\Theta) := \max_{1\le i,j\le N} |\theta_i - \theta_j|,$$

$$D_{\mathbb{T}}(\Theta) := \textit{The length of smallest arc containing } \left\{e^{\mathrm{i}\theta_1}, \cdots, e^{\mathrm{i}\theta_N}\right\}.$$

Remark 3.1.1. *1. The configuration Θ exhibits a heading angle alignment if and only if $D_{\mathbb{T}}(\Theta) \to 0$ as $t \to \infty$. Therefore, the diameter $D_{\mathbb{T}}$ gives another simple description of the heading angle alignment.*

2. *For any angle configuration* Θ, *we have*

$$D(\Theta) \le \pi \Longrightarrow D(\Theta) = D_{\mathbb{T}}(\Theta).$$

Therefore, the convergence of $D(\Theta) \to 0$ *implies the heading angle alignment of* Θ, *but the converse is not true in general since the diameter* $D(\Theta)$ *depends on the* 2π *translation of* $\{\theta_i\}_{i=1}^N$, *while the heading angle alignment does not.*

Finally, we close this section with the previous flocking estimate on the deterministic J-K model (1.0.4) in [45].

Theorem 3.1.1 ([45])**.** *Let* $\{(x_i, \theta_i)\}_{i=1}^N$ *be a solution to the equation* (1.0.4) *with real analytic communication weight* ψ, *where the initial configuration* $(X_0, \Theta_0) = \{(x_{i0}, \theta_{i0})\}_{i=1}^N$ *satisfies*

$$0 < D(\Theta_0) < \min\left\{\pi, \frac{\kappa \sin D(\Theta_0)}{2D(\Theta_0)} \int_{D(X_0)}^{\infty} \psi(s)ds\right\}.$$

Then, there exists a positive constant D^∞ *such that*

$$\begin{aligned} &(i)\ \sup_{0\le t<\infty} D(X(t)) \le D^\infty, \\ &(ii)\ D(\Theta_0)e^{-\kappa\psi(0)t} \le D(\Theta(t)) \le D(\Theta_{in})e^{-\frac{\kappa \sin D(\Theta_0)}{D(\Theta_0)}\psi(D^\infty)t}. \end{aligned} \tag{3.1.6}$$

3.2 Emergence of flocking for the deterministic J-K model

We now provide a detailed flocking estimate of the deterministic J-K system (1.0.4). We begin with an elementary lemma to be used in Section 3.2 and Section 3.3.

Lemma 3.2.1 ([50])**.** *Let* $\Theta = (\theta^1, \cdots, \theta^N) \in \mathbb{R}^N$ *be a heading angle configuration satisfying*

$$D(\Theta) \le 2\pi.$$

If there exists a constant $\delta \ge 0$ *and indices* $m, l, M \in \{1, \cdots, N\}$ *satisfying*

$$\theta^m - \delta \le \theta^l \le \theta^M + \delta,$$

we have

$$\sin(\theta^M - \theta^m) - \sin(\theta^M - \theta^l) - \sin(\theta^l - \theta^m) \le 2\delta. \tag{3.2.1}$$

Proof. For notational simplicity, we set

$$\theta^{ij} := \theta^i - \theta^j, \quad 1 \le i, j \le N.$$

Then we can write our target function

$$\sin(\theta^M - \theta^m) - \sin(\theta^M - \theta^l) - \sin(\theta^l - \theta^m)$$

to multiplicative form

$$\begin{aligned}
&\sin\theta^{Mm} - \sin\theta^{Ml} - \sin\theta^{lm} \\
&\quad = 2\sin\frac{\theta^{Mm}}{2}\left(\cos\frac{\theta^{Mm}}{2} - \cos\frac{\theta^{Ml} - \theta^{lm}}{2}\right) = -4\sin\frac{\theta^{Mm}}{2}\sin\frac{\theta^{Ml}}{2}\sin\frac{\theta^{lm}}{2}.
\end{aligned}$$

Now, consider the following trichotomy for $\theta^\ell, \theta^M, \theta^m$:

$$(1)\ \theta^l \le \theta^m (\le \theta^l + \delta) \quad (2)\ \theta^M \le \theta^l (\le \theta^M + \delta) \quad (3)\ \theta^m < \theta^l < \theta^M.$$

For the first two cases, we have

$$\text{either } |\theta^{lm}| \le \delta \quad \text{or} \quad |\theta^{Ml}| \le \delta.$$

This yields

$$\sin\theta^{Mm} - \sin\theta^{Ml} - \sin\theta^{lm} = -4\sin\frac{\theta^{Mm}}{2}\sin\frac{\theta^{Ml}}{2}\sin\frac{\theta^{lm}}{2} \le 4\cdot 1\cdot 1\cdot \sin\frac{\delta}{2} \le 2\delta.$$

On the other hand, for the third case, since $\frac{\theta^{Mm}}{2}$, $\frac{\theta^{Ml}}{2}$ and $\frac{\theta^{lm}}{2}$ are all contained in $[0, \pi]$ and therefore their sine values are all nonnegative. Thus, we can immediately see that (4.2.6) holds.

□

Remark 3.2.1. *For the case $\delta = 0$, the inequality* (3.2.1) *is equivalent to the trigonometric inequality for arbitrary triangle in a plane. The equality holds if and only if*

$$\theta^{Mm} \in \{0, 2\pi\} \quad \textit{or} \quad \theta^{Ml} \in \{0, 2\pi\} \quad \textit{or} \quad \theta^{lm} \in \{0, 2\pi\}.$$

We now review the non-increasing property of the diameter $D(\Theta)$ for the system (1.0.4). In [45], the communication weight function ψ is assumed to be real analytic, and therefore it is possible to find a sequence of times $\{\tau_k\}_{k\geq 0}$ so that the maximum and minimum angles among $\{\theta_i\}_{i=1}^N$ are uniquely determined in each time interval (τ_{k-1}, τ_k). This means that $\max_i \theta_i$ (resp. $\min_i \theta_i$) is piecewise C^1 and continuous functions, and the derivative is nonpositive (resp. nonnegative) as long as $D(\Theta) \leq \pi$. However, if ψ is assumed to be a generic Lipschitz continous function, we cannot simply use this argument to obtain the non-increasingness of $D(\Theta)$. We here provide a modified proof of the contractivity of diameter $D(\Theta)$ which can be applied to any Lipschitz function ψ.

Lemma 3.2.2. *Let $\{(x_j, \theta_j)\}_{j=1}^N$ be a classical solution to* (1.0.4) *with initial data $(X_0, \Theta_0) = \{(x_{j0}, \theta_{j0})\}_{j=1}^N$, and suppose that (X_0, Θ_0) satisfies*

$$D(\Theta_0) < \pi.$$

Then, the diameter $D(\Theta)$ is non-increasing in t :

$$D(\Theta(t)) \leq D(\Theta_0) < \pi, \quad \forall t \geq 0.$$

Proof. For every $\varepsilon \in (0, \pi - D(\Theta_0))$, consider a set $\mathcal{S}_\varepsilon$ denoted by

$$\mathcal{S}_\varepsilon := \{t > 0 : D(\Theta(t)) \geq D(\Theta_0) + \varepsilon\},$$

and assume that $\mathcal{S}_\varepsilon$ is nonempty. Then, since $D(\Theta(t))$ is continuous, the minimum $t_\varepsilon := \min \mathcal{S}_\varepsilon$ has to be finite and strictly positive.

Now for simplicity, we write

$$\theta_m := \min_k \theta_k, \quad \theta_M := \max_k \theta_k,$$

and

$$I_m(t) := \{i : \theta_i(t) = \theta_m(t)\}, \quad I_M(t) := \{i : \theta_i(t) = \theta_M(t)\}.$$

If $D(\Theta_0) = 0$, one can use the uniqueness of solution to (1.0.4) to deduce

$$\theta_{i0} = \theta_{j0} \quad \text{and} \quad \theta_i(t) \equiv \theta_{i0} \quad \forall\ i, j = 1, \cdots, N, \quad t > 0,$$

which contradicts to $\mathcal{S}_\varepsilon \neq \emptyset$.

For the case $D(\Theta_0) > 0$, we choose an arbitrary $(i,j) \in I_m(t_\varepsilon) \times I_M(t_\varepsilon)$ and obtain

$$\begin{aligned}
&\dot\theta_j(t_\varepsilon) - \dot\theta_i(t_\varepsilon) \\
&= \frac{\kappa}{N}\sum_{k=1}^N \psi(\|x_k(t_\varepsilon) - x_j(t_\varepsilon)\|)\sin(\theta_k(t_\varepsilon) - \theta_j(t_\varepsilon)) \\
&\qquad - \frac{\kappa}{N}\sum_{k=1}^N \psi(\|x_k(t_\varepsilon) - x_i(t_\varepsilon)\|)\sin(\theta_k(t_\varepsilon) - \theta_i(t_\varepsilon)) \\
&\le -\frac{\kappa}{N}\sum_{k=1}^N \psi(D(X(t_\varepsilon)))\left(\sin(\theta_j(t_\varepsilon) - \theta_k(t_\varepsilon)) + \sin(\theta_k(t_\varepsilon) - \theta_i(t_\varepsilon))\right) \\
&\le -\kappa\psi(D(X(t_\varepsilon)))\sin(\theta_j(t_\varepsilon) - \theta_i(t_\varepsilon)) \\
&= -\kappa\psi(D(X(t_\varepsilon)))\sin(D(\Theta_0) + \varepsilon),
\end{aligned} \tag{3.2.2}$$

where we used the non-increasingness of ψ and Lemma 3.2.1. Therefore, at time t_ε, we have $\dot\theta_j(t_\varepsilon) - \dot\theta_i(t_\varepsilon) < 0$, and hence

$$D(\Theta(t_\varepsilon - \delta)) \ge \theta_j(t_\varepsilon - \delta) - \theta_i(t_\varepsilon - \delta) > \theta_j(t_\varepsilon) - \theta_i(t_\varepsilon) = D(\Theta(t_\varepsilon)) \ge D(\Theta_0) + \varepsilon,$$

for some $\delta \in (0, t_\varepsilon)$. This contradicts to the minimality of t_ε, and we conclude $\mathcal{S}_\varepsilon = \emptyset$ for all $\varepsilon > 0$, which is the desired result. $\square$

In [45], the Lyapunov type funtionals

$$\mathcal{H}_\pm(X,\Theta) := 2D(\Theta) + \frac{\kappa \sin D(\Theta_0)}{D(\Theta_0)}\int_0^{D(X)} \psi(s)ds$$

for the deterministic J-K system (1.0.4) is considered, and presented the non-increasingness of $\mathcal{H}_\pm$ for initial configuration (X_0, Θ_0) with $D(\Theta_0) < \pi$ and real analytic ψ. we here define a similar Lyapunov functionals and study its non-increasing property for generic Lipschitz function ψ.

Lemma 3.2.3. *Let $\{(x_j, \theta_j)\}_{j=1}^N$ be a classical solution to* (1.0.4) *with initial data $(X_0, \Theta_0) = \{(x_{j0}, \theta_{j0})\}_{j=1}^N$, and suppose that (X_0, Θ_0) satisfies*

$$D(\Theta_0) < \pi.$$

Then, the functional

$$\mathcal{L}(X,\Theta;c) := 2\log\tan\left(\frac{\pi + D(\Theta)}{4}\right) + c\kappa\int_0^{D(X)}\psi(s)ds$$

is non-increasing for every $|c| \le 1$.

Proof. It suffices to prove the non-increasing property for every $|c| < 1$. For simplicity, we define an auxiliary functional

$$\mathcal{L}_{ij}^{k\ell}(X,\Theta;c) := 2\log\tan\left(\frac{\pi + \theta_j - \theta_i}{4}\right) + c\kappa\int_0^{\|x_k - x_\ell\|}\psi(s)ds.$$

Similar to the previous Lemma, we choose an arbitrary positive number ε and consider a set $\mathcal{G}_\varepsilon$ denoted by

$$\mathcal{G}_\varepsilon := \left\{t > 0 : \mathcal{L}(X(t),\Theta(t);c) \ge \mathcal{L}(X_0,\Theta_0;c) + \varepsilon\right\}.$$

Then, we use the notion θ_m, θ_M and $I_m(t), I_M(t)$ in Lemma 3.2.2 and define an index pair set $J(t)$ as

$$J(t) := \left\{(k,\ell) : \|x_k(t) - x_\ell(t)\| = D(X(t))\right\}.$$

Now, if $\mathcal{G}_\varepsilon$ is nonempty, the minimum $\tau_\varepsilon := \min \mathcal{G}_\varepsilon$ has to be finite and strictly positive, and in particular, $D(\Theta_0) > 0$. Moreover, for any $(i,j) \in I_m(\tau_\varepsilon) \times I_M(\tau_\varepsilon)$ and $(k,\ell) \in J(\tau_\varepsilon)$, we have

$$\mathcal{L}(X(\tau_\varepsilon),\Theta(\tau_\varepsilon);c) = \mathcal{L}_{ij}^{k\ell}(X(\tau_\varepsilon),\Theta(\tau_\varepsilon);c) \tag{3.2.3}$$

and at time $t = \tau_\varepsilon$,

$$\begin{aligned}
\left.\frac{d\mathcal{L}_{ij}^{k\ell}}{dt}\right|_{t=\tau_\varepsilon} &= \frac{\dot\theta_j - \dot\theta_i}{\cos\left(\frac{\theta_j - \theta_i}{2}\right)} + c\kappa\psi(D(X)) \cdot \left.\frac{d}{dt}\right|_{t=\tau_\varepsilon}\|x_k - x_\ell\| \\
&\le \kappa\psi(D(X))\left(-2\sin\left(\frac{\theta_j - \theta_i}{2}\right) + c\left.\frac{d}{dt}\right|_{t=\tau_\varepsilon}\|x_k - x_\ell\|\right)
\end{aligned}$$

$$
\begin{aligned}
&= \kappa\psi(D(X))\left(-2\sin\left(\frac{\theta_j-\theta_i}{2}\right)+c\frac{(x_k-x_\ell)\cdot(\dot{x}_k-\dot{x}_\ell)}{\|x_k-x_\ell\|}\right)\\
&\le \kappa\psi(D(X))\left(-2\sin\left(\frac{\theta_j-\theta_i}{2}\right)+c\|\dot{x}_k-\dot{x}_\ell\|\right)\\
&= \kappa\psi(D(X))\left(-2\sin\left(\frac{\theta_j-\theta_i}{2}\right)+c|e^{\mathrm{i}\theta_k}-e^{\mathrm{i}\theta_\ell}|\right)\\
&\le -2\kappa(1-c)\psi(D(X))\sin\left(\frac{D(\Theta)}{2}\right),
\end{aligned}
$$

where we used the upper bouned estimate of $\dot{\theta}_j-\dot{\theta}_i$ of (3.2.2) in the first inequality. That is to say, we have $\frac{d\mathcal{L}_{ij}^{k\ell}}{dt}<0$ at time $t=\tau_\varepsilon$ and therefore

$$
\begin{aligned}
\mathcal{L}(X(\tau_\varepsilon-\delta),\Theta(\tau_\varepsilon-\delta);c) &\ge \mathcal{L}_{ij}^{k\ell}(X(\tau_\varepsilon-\delta),\Theta(\tau_\varepsilon-\delta);c)\\
&> \mathcal{L}_{ij}^{k\ell}(X(\tau_\varepsilon),\Theta(\tau_\varepsilon);c)\\
&= \mathcal{L}(X(\tau_\varepsilon),\Theta(\tau_\varepsilon);c) \ge \mathcal{L}(X_0,\Theta_0;c)+\varepsilon,
\end{aligned}
$$

for some small $\delta>0$. Since this contradicts to the minimality of τ_ε, we deduce $\mathcal{G}_\varepsilon=\emptyset$ and obtain the desired result. $\square$

Finally, once we have the Lyapunov functional $\mathcal{L}$, we can deduce the following result.

Theorem 3.2.1. *Let $\{(x_j,\theta_j)\}_{j=1}^N$ be a classical solution to* (1.0.4) *with initial data $(X_0,\Theta_0)=\{(x_{j0},\theta_{j0})\}_{j=1}^N$, and suppose that (X_0,Θ_0) satisfies*

$$
0<D_{\mathbb{T}}(\Theta_0)<4\arctan\left(e^{\kappa\int_{D(X_0)}^\infty \psi(s)ds}\right)-\pi. \tag{3.2.4}
$$

Then, there exists a positive constant D^∞ such that

(*i*) $\displaystyle\sup_{0\le t<\infty} D(X(t))\le D^\infty$,

(*ii*) $\displaystyle e^{-\kappa\psi(0)t}\tan\frac{D_{\mathbb{T}}(\Theta_0)}{2}\le\tan\frac{D_{\mathbb{T}}(\Theta(t))}{2}\le e^{-\kappa\psi(D^\infty)t}\tan\frac{D_{\mathbb{T}}(\Theta_0)}{2}.$

Proof. Although the proof is similar to [45], we here present the proof for the reader's conveneince.

(i) First, adding 2π integer times to some θ_i's if necessary, we may assume that $D(\Theta_0) = D_{\mathbb{T}}(\Theta_0) < \pi$ without loss of generality. Then, the condition (3.2.4) ensures the existsence of a real number D^∞ satisfying

$$2 \log \tan\left(\frac{\pi + D(\Theta_0)}{4}\right) = \kappa \int_{D(X_0)}^{D^\infty} \psi(s) ds.$$

Then, we use the non-increasing property of $\mathcal{L}(X, \Theta, 1)$ to obtain

$$\kappa \int_0^{D(X(t))} \psi(s) ds \le \mathcal{L}(X(t), \Theta(t), 1) \le \mathcal{L}(X_0, \Theta_0, 1) = \kappa \int_0^{D^\infty} \psi(s) ds,$$

which is the desired result.

(ii) Since $D(X) \le D^\infty$ and ψ is nonincreasing, we have

$$\psi(D^\infty) \le \psi(\|x_i(t) - x_j(t)\|) \le \psi(0), \quad \forall\, t \ge 0,\ i, j = 1, \cdots, N.$$

Now, for every $i, j = 1, \cdots, N$, and $c > 1$, we define the functions g_{ij} and h_{ij} as

$$g_{ij}(t) := e^{c\kappa\psi(0)t} \tan \frac{\theta_j(t) - \theta_i(t)}{2}, \quad h_{ij}(t) := e^{c\kappa\psi(D^\infty)t} \tan \frac{\theta_j(t) - \theta_i(t)}{2},$$

and

$$g(t) := \max_{1 \le i,j \le N} g_{ij}(t), \quad h(t) := \max_{1 \le i,j \le N} h_{ij}(t).$$

For any $t > 0$ and $(i, j) \in I_m(t) \times I_M(t)$, we use (3.2.2) to obtain

$$\begin{aligned} g_{ij}(t) = g(t), \quad \frac{dg_{ij}}{dt} &= e^{c\kappa\psi(0)t} \left(c\kappa\psi(0) \tan \frac{\theta_j - \theta_i}{2} + \frac{\dot\theta_j - \dot\theta_i}{2 \cos^2 \frac{\theta_j - \theta_i}{2}} \right) \\ &\ge e^{c\kappa\psi(0)t} \tan \frac{\theta_j - \theta_i}{2} (c-1)\kappa\psi(0) > 0, \\ h_{ij}(t) = h(t), \quad \frac{dh_{ij}}{dt} &= e^{c\kappa\psi(D^\infty)t} \left(c\kappa\psi(D^\infty) \tan \frac{\theta_j - \theta_i}{2} + \frac{\dot\theta_j - \dot\theta_i}{2 \cos^2 \frac{\theta_j - \theta_i}{2}} \right) \\ &\le e^{c\kappa\psi(D^\infty)t} \tan \frac{\theta_j - \theta_i}{2} (c-1)\kappa\psi(D^\infty) < 0, \end{aligned}$$

provided that $D(\Theta_0) > 0$. Then, we use the $\mathcal{S}_\varepsilon - t_\varepsilon$ argument as in Lemma 3.2.2 and Lemma 3.2.3 to deduce the monotone increasing of $g(t)$ and monotone decreasing of $h(t)$. □

3.3 The stochastic persistency of the additive noise J-K model

We provide a lower-bound estimate of the probability of sample path to stay close to the flocking state for a certain finite time interval.

3.3.1 Basic sample path estimates

In this subsection, we recall two preparatory estimates in relation with the stochastic process.

Lemma 3.3.1. *Let B_t be the standard one-dimensional Brownian motion. Then, the following assertions hold:*

1. *(Andre's reflection Principle): for any time $T > 0$ and positive number $a > 0$, we have*

$$\mathbb{P}\Big\{ \sup_{0 \le t \le T} B_t \ge a \Big\} = 2\mathbb{P}\{B_T \ge a\} \le \sqrt{\frac{2T}{\pi a^2}} e^{-\frac{a^2}{2T}}.$$

2. *(Bounded Ornstein-Uhlenbeck (O-U) process [10]): there exist $c_0, r_0 > 0$ such that if*

$$r\left(\frac{\sqrt{\nu}h}{\sigma}, \nu T\right) := \frac{\sigma}{\sqrt{\nu}h} + \nu T e^{-c_0 \nu \frac{h^2}{\sigma^2}} \le r_0,$$

we have

$$\mathbb{P}\left\{ \sup_{0 \le t \le T} \sqrt{2} \int_0^t e^{-\nu(t-s)} dB_s \ge \frac{h}{\sigma} \right\} \le A e^{-\frac{\nu h^2}{2\sigma^2}},$$

where $A = A\left(\frac{h}{\sigma}, \nu, T\right)$ has a following order with respect to $\frac{h}{\sigma}, \nu, T$:

$$A = \sqrt{\frac{2\nu^3 T^2 h^2}{\pi \sigma^2}} \left[1 + \mathcal{O}\left(r\left(\frac{\sqrt{\nu}h}{\sigma}, \nu T\right) + \frac{1}{\nu} + \frac{1}{\nu T} \log\left(1 + \frac{\sqrt{\nu}h}{\sigma}\right) \right) \right].$$

3.3.2 Relaxed first collision-time

Recall that the additive noise Justh-Krishnaprasad model (3.0.1) was given as:

$$\begin{aligned} dx_t^j &= (\cos\theta_t^j, \sin\theta_t^j)\ dt, \quad t > 0,\ j = 1, \cdots, N, \\ d\theta_t^j &= \frac{\kappa}{N}\sum_{k=1}^{N} \psi(\|x_t^k - x_t^j\|)\sin(\theta_t^k - \theta_t^j)dt + \sqrt{2\sigma}dB_t^j. \end{aligned}$$

In the deterministic model (1.0.4), once the communication weight function ψ is analytic, then the collisions between two heading angles occur only finitely in any finite-time interval. Thus, the diameter $D(\Theta_t)$ can be described in terms of maximum and minimum phases $\theta^M(t)$ and $\theta^m(t)$ which are piecewise analytic.

However, in the stochastic model (3.0.1), the zero set of relative heading angles $\theta^i - \theta^j$ can be infinite in a finite time interval due to the lack of regularity. Thus, we need to relax the concept of the maximum and minimum heading angles. We adopt the technique of "*relaxed first-collision time*" introduced in [41] and estimate the dynamics of $D(\Theta_t)$ until the collision time.

For a given $\delta > 0$ and initial configuration $(X_{in}, \Theta_{in}) \in \mathbb{R}^{2N} \times \mathbb{R}^N$, we choose one pair of indices (M_0, m_0) and define the relaxed first collision-time $\tau^0(\delta) := \tau^0(\delta; X_{in}, \Theta_{in})$ for the time-dependent interval $(\theta_t^{m_0} - \delta, \theta_t^{M_0} + \delta)$ as

$$\begin{aligned} &\theta_{in}^{m_0} := \min_{1\le j\le N} \theta_{in}^j, \quad \theta_{in}^{M_0} := \max_{1\le j\le N} \theta_{in}^j, \quad \text{and} \\ &\tau^0(\delta; X_{in}, \Theta_{in}) := \inf\left\{t > 0 : \theta_t^i \notin (\theta_t^{m_0} - \delta, \theta_t^{M_0} + \delta) \quad \text{for some } i\right\}. \end{aligned} \tag{3.3.1}$$

From the definition of $\tau^0(\delta)$, the phases $\theta_t^{m_0}$ and $\theta_t^{M_0}$ may not be the minimum or maximum of $\{\theta_t^1, \cdots, \theta_t^N\}$ for some $t < \tau^0(\delta)$. Instead, for $t < \tau^0(\delta)$, we have the following estimates on the diameter of θ_t^j:

$$\max_{j,k} |\theta_t^k - \theta_t^j| \le \theta_t^{M_0} - \theta_t^{m_0} + 2\delta.$$

Therefore, we can use the alternative process $\theta_t^{M_0} - \theta_t^{m_0} + 2\delta$ to bound $D(\Theta)$ until $\tau^0(\delta)$.

3.3.3 Estimate on the relaxed first collision-time

We present a quantitative estimate on $\tau^0(\delta)$. Even in the deterministic case, $\tau^0(\delta)$ may not be infinity because of the δ bump. Hence, we estimate the probability that $\tau^0(\delta)$ is less than a given small constant value. For this, we assume that the communication weight function ψ is Lipschitz continuous and strictly positive such that there exist positive constants ψ_m and ψ_M satisfying

$$0 < \psi_m \le \psi(\|x-y\|) \le \psi_M, \quad \forall\, x, y \in \mathbb{R}^2.$$

Then, define

$$C_\delta := (1+\sin\delta)\psi_M - \cos\delta\psi_m \quad \text{and} \quad T_\delta := \frac{\delta}{2\kappa C_\delta}. \tag{3.3.2}$$

Note that for $0 < \delta \ll 1$, we have

$$\lim_{\delta\to 0+} C_\delta = \psi_M - \psi_m > 0, \quad \lim_{\delta\to 0+} \frac{T_\delta}{\delta} = \frac{1}{2\kappa(\psi_M - \psi_m)}.$$

Now, let (X_t, Θ_t) be a solution process of (3.0.1) issued from the initial state (X_{in}, Θ_{in}). Below, we provide an estimation on the following probability:

$$\mathbb{P}\Big\{\tau^0(\delta; X_{in}, \Theta_{in}) < T_\delta\Big\}.$$

Note that the defining relation (3.3.1) implies

$$\begin{aligned} &\tau^0(\delta; X_{in}, \Theta_{in}) < T_\delta \\ &\qquad \Longleftrightarrow \quad \exists\, j \in \{1, \cdots, N\},\ t < T_\delta \quad \text{such that} \\ &\qquad \text{either } \theta_t^j \le \theta_t^{m_0} - \delta \quad \text{or} \quad \theta_t^j \ge \theta_t^{M_0} + \delta. \end{aligned} \tag{3.3.3}$$

Hence, we introduce the collision time of each pair of angles: for $\delta \ge 0$ and a configuration (X_{in}, Θ_{in}),

$$\tau^{ij}(\delta; X_{in}, \Theta_{in}) := \inf\left\{t > 0 : \theta_t^i + \delta \le \theta_t^j, (X_0, \Theta_0) = (X_{in}, \Theta_{in})\right\}. \tag{3.3.4}$$

Then, the probability for the event (3.3.3) can be expressed in terms of (3.3.4):

$$\begin{aligned} &\mathbb{P}\{\tau^0(\delta; X_{in}, \Theta_{in}) < T_\delta\} \\ &\quad = \mathbb{P}\{\exists\, j \text{ s.t. } \inf_{t<T_\delta}(\theta_t^j - \theta_t^{m_0}) \le -\delta \text{ or } \inf_{t<T_\delta}(\theta_t^{M_0} - \theta_t^j) \le -\delta\} \\ &\quad \le \mathbb{P}\{\exists\, j : \tau^{jm_0}(\delta; X_{in}, \Theta_{in}) < T_\delta\} + \mathbb{P}\{\exists\, j : \tau^{M_0 j}(\delta; X_{in}, \Theta_{in}) < T_\delta\}. \end{aligned} \tag{3.3.5}$$

In the following lemma, we estimate the above probabilities quantitatively.

Lemma 3.3.2. *For any $\delta \in (0, \frac{\pi}{2})$, the following estimates hold.*

$$(i)\ \mathbb{P}\{\tau^{jm_0}(\delta; X_{in}, \Theta_{in}) < T_\delta\} \le 8\sqrt{\frac{\sigma}{\pi\kappa\delta C_\delta}}e^{-\frac{\kappa\delta C_\delta}{16\sigma}}.$$

$$(ii)\ \mathbb{P}\{\tau^{M_0 j}(\delta; X_{in}, \Theta_{in}) < T_\delta\} \le 8\sqrt{\frac{\sigma}{\pi\kappa\delta C_\delta}}e^{-\frac{\kappa\delta C_\delta}{16\sigma}}.$$

Proof. Since the derivation of the estimate (ii) will be similar to that of (i), we only consider the estimate (i) below. We split the proof into three steps.

• Step A: For each j, from the sample space Ω, consider a continuous sample path $\theta_t^{jm_0}(\omega) := \theta_t^j(\omega) - \theta_t^{m_0}(\omega)$ for $\omega \in \Omega$, where

$$\omega \in \left\{\inf_{t<T_\delta}(\theta_t^j - \theta_t^{m_0}) \le -\delta\right\} = \{\tau^{jm_0}(\delta; X_{in}, \Theta_{in}) < T_\delta\}.$$

Since each θ^j has a continuous path, we know

$$\tau^{jm_0}(0; X_{in}, \Theta_{in}) < \tau^{jm_0}(\delta; X_{in}, \Theta_{in}).$$

Thus, we may change the initial time into $\tau^{jm_0}(0)$ in the following way: For given initial data (X_{in}, Θ_{in}) and $\delta > 0$,

$$\begin{aligned}\left\{\omega : \inf_{t<T_\delta}\theta_t^{jm_0}(\omega) \le -\delta\right\} &= \left\{\omega : \inf_{\tau^{jm_0}(0)<t<T_\delta}\theta_t^{jm_0}(\omega) \le -\delta\right\}\\ &\subset \left\{\omega : \inf_{\tau^{jm_0}(0)<t<T_\delta+\tau^{jm_0}(0)}\theta_t^{jm_0}(\omega) \le -\delta\right\}\\ &= \left\{\omega : \inf_{0<t<T_\delta}\left(\beta_t^j(\omega) - \beta_t^{m_0}(\omega)\right) \le -\delta\right\},\end{aligned}$$

where $\beta_t^i(\omega) := \theta^i_{t+\tau^{jm_0}(0)}(\omega)$ for all $i = 1, \cdots, N$ and $t \ge 0$.

Since we do not have enough information on β_0^i, we do the worst-case analysis. Let Λ^{ij} be the set of all initial configuration $(\tilde{X}_{in}, \tilde{\Theta}_{in})$ satisfying $\tilde{\theta}_{in}^i = \tilde{\theta}_{in}^j$. Then, we consider the collision-time $\tilde{\tau}^{ij} = \tilde{\tau}^{ij}(\delta, X_{in}, \Theta_{in})$ for $(X_{in}, \Theta_{in}) \in \Lambda^{ij}$ as

$$\tilde{\tau}^{ij}(\delta; X_{in}, \Theta_{in}) := \inf\left\{t > 0 : |\theta_t^i - \theta_t^j| \ge \delta,\ (X_0, \Theta_0) = (X_{in}, \Theta_{in})\right\}.$$

Since $\beta_0^{jm_0} = 0$ and the stochastic process β_t satisfies the same SDE (3.0.1) as θ_t, we have

$$\begin{aligned}
&\mathbb{P}\{\tau^{jm_0}(\delta; X_{in}, \Theta_{in}) < T_\delta\} \\
&\qquad = \mathbb{P}\left\{\omega : \inf_{t<T_\delta} \theta_t^{jm_0}(\omega) \le -\delta\right\} \\
&\qquad \le \mathbb{P}\left\{\omega : \inf_{t<T_\delta} \beta_t^{jm_0}(\omega) \le -\delta\right\} \\
&\qquad \le \sup_{(\tilde{X}_{in}, \tilde{\Theta}_{in}) \in \Lambda^{jm_0}} \mathbb{P}\{\tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) < T_\delta\},
\end{aligned} \tag{3.3.6}$$

where we used the strong Markov property of (X_t, Θ_t) in the last inequality (see Appendix A.4).

• Step B: Consider the initial configuration $(\tilde{X}_{in}, \tilde{\Theta}_{in}) \in \Lambda^{jm_0}$, and for simplicity, we set

$$\tilde{\psi}_t^{ij} := \psi(\|\tilde{x}_t^i - \tilde{x}_t^j\|) \quad \text{and} \quad \tilde{\theta}_t^{ij} := \tilde{\theta}_t^i - \tilde{\theta}_t^j, \quad i, j = 1, \cdots, N.$$

We claim: for $t < \tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) \wedge T_\delta$,

$$\begin{aligned}
\tilde{\theta}_t^j - \tilde{\theta}_t^{m_0} &\ge -\frac{\delta}{2} + \sqrt{2\sigma}(B_t^j - B_t^{m_0}), \\
\tilde{\theta}_t^j - \tilde{\theta}_t^{m_0} &\le \frac{\delta}{2} + \sqrt{2\sigma}(B_t^j - B_t^{m_0}).
\end{aligned} \tag{3.3.7}$$

Proof of (3.3.7): Note that for $t < \tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in})$, we use the trigonometric identity:

$$\sin \theta_t^{km_0} = \sin \tilde{\theta}_t^{kj} \cos \tilde{\theta}_t^{jm_0} + \cos \tilde{\theta}_t^{kj} \sin \tilde{\theta}_t^{jm_0}$$

to get

$$\begin{aligned}
d\tilde{\theta}_t^{jm_0} &= \frac{\kappa}{N} \sum_{k=1}^N \left(\tilde{\psi}_t^{kj} \sin \tilde{\theta}_t^{kj} - \tilde{\psi}_t^{km_0} \sin \tilde{\theta}_t^{km_0}\right) dt + \sqrt{2\sigma} d(B_t^j - B_t^{m_0}) \\
&= \frac{\kappa}{N} \sum_{k=1}^N \underbrace{\left((\tilde{\psi}_t^{kj} - \tilde{\psi}_t^{km_0} \cos \tilde{\theta}_t^{jm_0}) \sin \tilde{\theta}_t^{kj} - \tilde{\psi}_t^{km_0} \sin \tilde{\theta}_t^{jm_0} \cos \tilde{\theta}_t^{kj}\right)}_{=:\mathcal{J}_k^{jm_0}} dt \\
&\quad + \sqrt{2\sigma} d(B_t^j - B_t^{m_0}).
\end{aligned} \tag{3.3.8}$$

$\diamond$ Case A (Derivation of $(3.3.7)_1$): To derive a worst lower bound for $\mathcal{J}$, we use

$$|\tilde{\theta}_t^j - \tilde{\theta}_t^{m_0}| \le \delta, \quad \text{for } t < \tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in})$$

and

$$\left|\tilde{\psi}_t^{kj} - \tilde{\psi}_t^{km_0} \cos\tilde{\theta}_t^{jm_0}\right| \le \psi_M - \psi_m \cos\delta, \quad \left|\tilde{\psi}_t^{km_0} \sin\tilde{\theta}_t^{jm_0} \cos\tilde{\theta}_t^{kj}\right| \le \psi_M \sin\delta,$$

to see

$$\begin{aligned}\mathcal{J}_k^{jm_0} &= (\tilde{\psi}_t^{kj} - \tilde{\psi}_t^{km_0} \cos\tilde{\theta}_t^{jm_0}) \sin\tilde{\theta}_t^{kj} - \tilde{\psi}_t^{km_0} \sin\tilde{\theta}_t^{jm_0} \cos\tilde{\theta}_t^{kj} \\ &\ge -\Big(\psi_M(1+\sin\delta) - \psi_m\cos\delta\Big) = -C_\delta.\end{aligned} \tag{3.3.9}$$

Therefore, we combine (3.3.8) and (3.3.9) to get

$$d\tilde{\theta}_t^{jm_0} \ge -\kappa C_\delta dt + \sqrt{2\sigma} d(B_t^j - B_t^{m_0}), \quad t < \tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}). \tag{3.3.10}$$

Next, we integrate the above inequality (3.3.10) using the relations:

$$\tilde{\theta}_0^j - \tilde{\theta}_0^{m_0} = 0 \quad \text{and} \quad \kappa C_\delta t \le \kappa C_\delta T_\delta = \frac{\delta}{2}, \quad \text{for } t < \tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) \wedge T_\delta$$

to get

$$\tilde{\theta}_t^j - \tilde{\theta}_t^{m_0} \ge -\frac{\delta}{2} + \sqrt{2\sigma}(B_t^j - B_t^{m_0}), \quad t < \tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) \wedge T_\delta. \tag{3.3.11}$$

$\diamond$ Case B (Estimate of $(3.3.7)_2$): Similar to Case A, we have

$$\tilde{\theta}_t^j - \tilde{\theta}_t^{m_0} \le \frac{\delta}{2} + \sqrt{2\sigma}(B_t^j - B_t^{m_0}), \quad t < \tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) \wedge T_\delta. \tag{3.3.12}$$

Finally, (3.3.11) and (3.3.12) yield the desired estimate (3.3.7).

• Step C: Next, we show the following inclusion relation:

$$\{\tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) < T_\delta\} \subset \left\{\sup_{t \le T_\delta} \left|\sqrt{2\sigma}(B_t^j - B_t^{m_0})\right| \ge \frac{\delta}{2}\right\}. \tag{3.3.13}$$

Suppose that there exist a sample point ω satisfying

$$\tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in})(\omega) < T_\delta,$$

and

$$\sup_{t\le T_\delta}\left|\sqrt{2\sigma}(B_t^j - B_t^{m_0})\right|(\omega) < \frac{\delta}{2}.$$

Then, we have

$$\begin{aligned}
&\sup_{t\le\tilde\tau^{jm_0}}\left|\tilde\theta_t^j - \tilde\theta_t^{m_0}\right| \\
&\quad\le \sup_{t\le\tilde\tau^{jm_0}}\left|\tilde\theta_t^j - \tilde\theta_t^{m_0} - \sqrt{2\sigma}(B_t^j - B_t^{m_0})\right| + \sup_{t\le\tilde\tau^{jm_0}}\left|\sqrt{2\sigma}(B_t^j - B_t^{m_0})\right| \\
&\quad\le \sup_{t\le\tilde\tau^{jm_0}\wedge T_\delta}\left|\tilde\theta_t^j - \tilde\theta_t^{m_0} - \sqrt{2\sigma}(B_t^j - B_t^{m_0})\right| + \sup_{t\le T_\delta}\left|\sqrt{2\sigma}(B_t^j - B_t^{m_0})\right| \\
&\quad< \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{aligned}$$

Since this gives a contradiction to the definition of $\tilde\tau^{jm_0}$, the inclusion relation (3.3.13) holds.

• Final step: Let $(\tilde X_{in}, \tilde\Theta_{in})$ be any initial configuration in Λ^{jm_0}. Then, we use the relation (3.3.13), Andre's reflection principle (see (i) in Lemma 3.3.1) and the relation

$$B_t^j - B_t^{m_0} = \sqrt{2}\tilde B_t \quad \text{for some standard Brownian motion } \tilde B_t$$

to obtain

$$\begin{aligned}
&\mathbb{P}\{\tilde\tau^{jm_0}(\delta;\tilde X_{in},\tilde\Theta_{in}) < T_\delta\} \le \mathbb{P}\left\{\sup_{t\le T_\delta}\left|\sqrt{2\sigma}(B_t^j - B_t^{m_0})\right| \ge \frac{\delta}{2}\right\} \\
&\quad= 2\mathbb{P}\left\{\sup_{t\le T_\delta}\sqrt{2\sigma}(B_t^j - B_t^{m_0}) \ge \frac{\delta}{2}\right\} = 2\mathbb{P}\left\{\sup_{t\le T_\delta}\tilde B_t \ge \frac{\delta}{4\sqrt{\sigma}}\right\} \\
&\quad\le 8\sqrt{\frac{2\sigma T_\delta}{\pi\delta^2}}e^{-\frac{\delta^2}{32\sigma T_\delta}} = 8\sqrt{\frac{\sigma}{\pi\kappa\delta C_\delta}}e^{-\frac{\kappa\delta C_\delta}{16\sigma}}.
\end{aligned} \tag{3.3.14}$$

Finally, the desired estimate follows from (3.3.5), (3.3.6) and (3.3.14). □

As a direct application of Lemma 3.3.2, we obtain the probability estimate on the event $\{\tau^0(\delta; X_{in},\Theta_{in}) < T_\delta\}$ as follows.

Proposition 3.3.1. *Suppose that for a positive number $\varepsilon > 0$, the system parameters κ, σ and control parameter δ satisfy*

$$\kappa > 0, \quad \sigma > 0, \quad \delta \in \left(0, \frac{\pi}{2}\right), \quad \sqrt{\frac{16\sigma}{\kappa\delta C_\delta}} < \varepsilon, \tag{3.3.15}$$

and let (X_t, Θ_t) be a stochastic J-K flow with the initial state (X_{in}, Θ_{in}). Then, for any small $\delta > 0$, we have

$$\mathbb{P}\{\tau^0(\delta; X_{in}, \Theta_{in}) < T_\delta\} \le \frac{4N}{\sqrt{\pi}} \varepsilon e^{-\varepsilon^{-2}}. \tag{3.3.16}$$

Proof. We combine (3.3.6)–(3.3.14) to deduce

$$\begin{aligned}
&\mathbb{P}\{\tau^0(\delta; X_{in}, \Theta_{in}) < T_\delta\} \\
&\le \sum_j \mathbb{P}\{\tilde{\tau}^{jm_0}(\delta; X_{in}, \Theta_{in}) < T_\delta\} + \sum_j \mathbb{P}\{\tilde{\tau}^{M_0 j}(\delta; X_{in}, \Theta_{in}) < T_\delta\} \\
&= \sum_j \left[\sup_{(\tilde{X}_{in}, \tilde{\Theta}_{in}) \in \Lambda^{jm_0}} \mathbb{P}\{\tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) < T_\delta\} \right] \\
&\qquad\qquad + \sum_j \left[\sup_{(\tilde{X}_{in}, \tilde{\Theta}_{in}) \in \Lambda^{Mj}} \mathbb{P}\{\tilde{\tau}^{Mj}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) < T_\delta\} \right] \\
&\le 16N \sqrt{\frac{\sigma}{\pi\kappa\delta C_\delta}} e^{-\frac{\kappa\delta C_\delta}{16\sigma}}.
\end{aligned}$$

Now, we use the relation (3.3.15) to get the desired estimate. $\square$

Remark 3.3.1. *1. For the constant $\psi(r) = \psi_\infty$, we can improve the estimate* (3.3.16) *as follows.*

$$\begin{aligned}
d\theta_t^{jm_0} &= \frac{\kappa\psi_\infty}{N} \sum_{k=1}^N \left(\sin\theta_t^{kj} - \sin\theta_t^{km_0}\right) dt + \sqrt{2\sigma} d(B_t^j - B_t^{m_0}) \\
&\ge -\kappa\psi_\infty \delta dt + \sqrt{2\sigma} d(B_t^j - B_t^{m_0}), \quad t < \tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}).
\end{aligned}$$

Therefore, for any $\delta > 0$, we have

$$
\begin{aligned}
&\mathbb{P}\left\{\tilde{\tau}^{jm_0}(\delta;\tilde{X}_{in},\tilde{\Theta}_{in}) < \frac{1}{2\kappa\psi_\infty}\right\} \le \mathbb{P}\left\{\sup_{t\le 1/(2\kappa\psi_\infty)}\left|\sqrt{2\sigma}(B_t^j - B_t^{m_0})\right| > \frac{\delta}{2}\right\}\\
&\quad = 2\mathbb{P}\left\{\sup_{t\le 1/(2\kappa\psi_\infty)}\sqrt{2\sigma}(B_t^j - B_t^{m_0}) > \frac{\delta}{2}\right\} \le 8\sqrt{\frac{\sigma}{\pi\kappa\psi_\infty\delta^2}}e^{-\frac{\kappa\psi_\infty\delta^2}{16\sigma}},\\
&\mathbb{P}\{\tau^0(\delta;X_0,\Theta_0) < T_\delta\} \le 16N\sqrt{\frac{\sigma}{\pi\kappa\psi_\infty\delta^2}}e^{-\frac{\kappa\psi_\infty\delta^2}{16\sigma}}.
\end{aligned}
$$

2. *The $N-$dependency of the result* (3.3.16) *comes from the definition of τ^0 and the additive noise in* (3.0.1). *Since $\tau^0 = \tau^0(\delta)$ is defined as the first collision time of θ_t^j to either $\theta_t^{m_0} - \delta$ or $\theta_t^{M_0} + \delta$ for at least one j, the probability of the event*

$$\{\tau^0 < T_\delta\}$$

can only be controlled by the $N-$times of probability for the collision event for each θ_t^j, unless we can relate the event of collsions for different θ^js. However, as the additive noises $\{dB_t^j\}_{j=1}^N$ are assumed to be independent to each other, we are not able to acheive this.

3.3.4 Description of main result

Below, we deduce the stochastic persistency of phases, which provides a nontrivial lower bound for the probability in which the system stays near the heading alignment state. Since the proof for our main result will be very lengthy, we first briefly discuss our main result and a strategy to prove, and then provide its detailed proof in the next subsection. Although we cannot expect the asymptotic heading angle alignment of the stochastic flow (X_t, Θ_t) as in Section 3.2, we analyze the first exit time of a process $D(\Theta_t)$ from finite interval $[0, D_\infty]$, where D_∞ is a given positive constant smaller than π. In fact, we need some time-dependent barrier function $L(s)$ which gives a sharper bound on $D(\Theta_t)$, i.e., $L(s) \le D_\infty$ for all $s \ge 0$. Then, under some sufficient framework, we estimate the probability of the event

$$\left\{D(\Theta_s) \text{ exceeds } L(s) \text{ at least once for } s \in [0,t]\right\},$$

which naturally induces the boundedness (by D_∞) and decreasing behavior of $D(\Theta_s)$.

Before we present the stochastic persistency estimate for (3.0.1), we introduce several notations including the barrier function $L(s)$: for positive constants δ and $D_\infty < \pi$, we set

$$\begin{aligned}
R_\infty &:= \frac{\sin D_\infty}{D_\infty}, \quad C_\delta := (1+\sin\delta)\psi_M - \cos\delta\psi_m, \quad T_\delta := \frac{\delta}{2\kappa C_\delta},\\
L(s) &:= (D(\Theta_0)+\delta)e^{-\kappa\psi_m R_\infty s} + \frac{2\delta\psi_M}{\psi_m R_\infty}(1-e^{-\kappa\psi_m R_\infty s}) + \delta\\
&\quad + 2\delta\sum_{r=1}^{\infty} e^{-\kappa\psi_m R_\infty (s-rT_\delta)}\chi_{(rT_\delta,\ \infty)},\\
P_\infty &:= A\left(\frac{\delta}{\sqrt{2\sigma}}, \kappa\psi_m R_\infty, T_\delta\right)e^{-\frac{\kappa\psi_m R_\infty \delta^2}{4\sigma}} + 16N\sqrt{\frac{\sigma}{\pi\delta\kappa C_\delta}}e^{-\frac{\delta\kappa C_\delta}{16\sigma}},
\end{aligned} \tag{3.3.17}$$

where $r = r\left(\frac{\sqrt{\nu}h}{\sigma}, \nu T\right)$ and $A = A\left(\frac{h}{\sigma}, \nu, T\right)$ are the functions defined in Lemma 3.3.1:

$$r\left(\frac{\sqrt{\nu}h}{\sigma}, \nu T\right) = \frac{\sigma}{\sqrt{\nu}h} + \nu T e^{-c_0\nu\frac{h^2}{\sigma^2}} \le r_0,$$

$$A = \sqrt{\frac{2\nu^3T^2h^2}{\pi\sigma^2}}\left[1 + \mathcal{O}\left(r\left(\frac{\sqrt{\nu}h}{\sigma}, \nu T\right) + \frac{1}{\nu} + \frac{1}{\nu T}\log\left(1+\frac{\sqrt{\nu}h}{\sigma}\right)\right)\right].$$

Now, we are ready to state our second main result on the emergent dynamics for (3.0.1).

Theorem 3.3.1. *Suppose that the initial data Θ_0 and system parameters κ, σ, δ satisfy the following relations:*

$$\begin{aligned}
&(i)\ \max\left\{D(\Theta_0)+2\delta,\ \frac{2\delta\psi_M}{\psi_m R_\infty} + 2\delta + \frac{4\delta}{1-e^{-\frac{\psi_m R_\infty\delta}{2C_\delta}}}\right\} < D_\infty < \pi,\\
&(ii)\ r\left(\delta\sqrt{\frac{\kappa\psi_m R_\infty}{\sigma}}, \kappa\psi_m R_\infty T_\delta\right) \le r_0,\\
&(iii)\ P_\infty < 1,
\end{aligned} \tag{3.3.18}$$

and let Θ_t *be a solution to* (3.0.1) *with the initial data* Θ_{in}*. Then, for any positive integer* $\ell \geq 1$*,*

$$\mathbb{P}\left\{\exists s < \ell T_\delta : D(\Theta_s) > L(s)\right\} \leq 1 - (1 - P_\infty)^\ell. \tag{3.3.19}$$

Proof. We use iterative methods using the time step T_δ, which is given by the collision time $\tau^0(\delta)$. In order to use induction, we split the proof into several steps. Here, we briefly sketch three main steps, and the detailed proof is given in next subsection:

- Step A: At $t = 0$, we choose the maximal and minimal values $\theta_0^{M_0}$ and $\theta_0^{m_0}$ among $\{\theta_t^1, \cdots, \theta_t^N\}$, and fix the indices M_0 and m_0. Then, for $t \in [0, \tau^0(\delta))$, we have

  $$D(\Theta_t) \leq \theta_t^{M_0} - \theta_t^{m_0} + 2\delta.$$

  We estimate the process $\theta_t^{M_0} - \theta_t^{m_0}$, which is expected to decrease in a high probability as in the deterministic model.

- Step B: Provided that $\tau^0(\delta) \geq T_\delta$, we reindex maximal and minimal indices of θ at time $t = T_\delta$ defined in (3.3.17) and do Step A again: We choose the maximum and minimum indices M_1 and m_1 at time T_δ, and estimate the process $D(\Theta_t)$ until $t = 2T_\delta$ by using θ^{M_1} and θ^{m_1}. By iterating this procedure, we may estimate the probability on the bounds of $D(\Theta_{(\ell+1)T_\delta})$ under proper assumptions on $D(\Theta_{\ell T_\delta})$.

- Step C: After estimating $D(\Theta_t)$ inductively, we will estimate the probability for the whole interval $[0, t]$. We split this probability into three parts. The first part is from the fluctuation of θ_t^j induced by noise, which can be estimated by Lemma 3.3.1 (ii). Secondly, we estimate $\{\tau(\delta) > T_\delta\}$ in Proposition 3.3.1 at each time $t = kT_\delta$. The last one is on the assumption of $\max_{j,k} |\theta_t^k - \theta_t^j| < D_\infty$, which will be treated by the assumptions (3.3.18). In conclusion, we build a recurrence inequality

  $$\mathbb{P}\left\{\tau_L \geq (\ell+1)T_\delta\right\} \geq (1 - P_\infty)\mathbb{P}\left\{\tau_L \geq \ell T_\delta\right\}, \tag{3.3.20}$$

for the stopping time τ_L which represents the left-hand side of (3.3.19),

$$\tau_L := \inf\{s > 0 : \ D(\Theta_s) \geq L(s)\}.$$

$\square$

Remark 3.3.2. *Note that the estimate* (3.3.19) *can be written as*

$$\mathbb{P}\left\{ \max_{0 \leq s \leq \ell T_\delta} D(\Theta_s) \leq L(s) \right\} > (1 - P_\infty)^\ell.$$

Thus, in a finite-time interval $[0, \ell T_\delta]$*, at least with the probability* $(1 - P_\infty)^\ell$*, the J-K particles* (3.0.1) *will stay in a region where* $D(\Theta)$ *is less than* $L(s)$*. Since the estimate becomes trivial as* $\ell \to \infty$*, our estimate* (3.3.19) *provides a useful information only in a finite-time interval. Moreover, from how we defined* P_∞ *in* (3.3.17)*, one can see that choosing the constant* δ *arbitrarily small makes* P_∞ *larger than* 1*. Therefore, in Theorem 3.3.1, it is more likely to consider* δ *as a fixed positive constant, and then* P_∞ *becomes small when* $\frac{\kappa}{\sigma}$ *is large.*

3.3.5 Proof of Theorem 3.3.1

From now on, we provide the detailed proof of Theorem 3.3.1.

Step A (Initial time-zone estimates)

For given initial data, we choose extremal indices M_0, m_0 as in (3.3.1). Then, we analyze the evolution of $\theta_t^{M_0 m_0} := \theta_t^{M_0} - \theta_t^{m_0}$ in time $t \in [0, T_\delta]$. For the condition on the half circle, we define the first hitting time of $\theta_t^{M_0 m_0} + 2\delta$ to D_∞ as follows:

$$\tau^0_{D_\infty}(\delta) := \tau^0_{D_\infty}(\delta, X_{in}, \Theta_{in}) := \inf\{t > 0 : \ \underbrace{\theta_t^{M_0 m_0} + 2\delta}_{\approx \max_{j,k} |\theta_t^k - \theta_t^j|} \geq D_\infty\}.$$

Note that for $t < \tau^0(\delta)$,

$$\theta_t^{m_0} - \delta \leq \theta_t^k \leq \theta_t^{M_0} + \delta, \quad k = 1, \cdots, N.$$

This yields

$$0 \le \theta_t^{km_0} + \delta, \qquad \theta_t^{kM_0} - \delta \le 0. \tag{3.3.21}$$

On the other hand, for $t < \tau_{D_\infty}^0(\delta)$,

$$\theta_t^{M_0 m_0} + 2\delta = (\theta_t^{M_0 k} + \delta) + (\theta_t^{km_0} + \delta) \le D_\infty. \tag{3.3.22}$$

Finally, we combine (3.3.21) and (3.3.22) to see that for $t < \tau^0(\delta) \wedge \tau_{D_\infty}^0(\delta)$,

$$0 \le \theta_t^{km_0} + \delta \le D_\infty < \pi \quad \text{and} \quad -\pi < -D_\infty \le \theta_t^{kM_0} - \delta \le 0, \quad k = 1, \cdots, N.$$

Now, we use the inequality $|\sin x - \sin y| \le |x - y|$ to see

$$|\sin\theta_t^{kM_0} - \sin(\theta_t^{kM_0} - \delta)| \le \delta, \quad |\sin\theta_t^{km_0} - \sin(\theta_t^{km_0} + \delta)| \le \delta. \tag{3.3.23}$$

Then, we use (3.3.23) to see that for $t < \tau^0(\delta) \wedge \tau_{D_\infty}^0(\delta)$,

$$\begin{aligned}
& d\theta_t^{M_0 m_0} \\
&= \frac{\kappa}{N}\sum_{k=1}^N \Big[\psi_t^{kM_0}\sin\theta_t^{kM_0} - \psi_t^{km_0}\sin\theta_t^{km_0}\Big]dt + \sqrt{2\sigma}d(B_t^{M_0} - B_t^{m_0}) \\
&\le \frac{\kappa}{N}\sum_{k=1}^N \Big[\psi_t^{kM_0}\Big(\sin(\theta_t^{kM_0} - \delta) + \delta\Big) - \psi_t^{km_0}\Big(\sin(\theta_t^{km_0} + \delta) - \delta\Big)\Big]dt \\
&\quad + \sqrt{2\sigma}d(B_t^{M_0} - B_t^{m_0}) \\
&= \frac{\kappa}{N}\sum_{k=1}^N \Big[-\psi_t^{kM_0}\Big(\sin(\theta_t^{M_0 k} + \delta) - \delta\Big) - \psi_t^{km_0}\Big(\sin(\theta_t^{km_0} + \delta) - \delta\Big)\Big]dt \\
&\quad + \sqrt{2\sigma}d(B_t^{M_0} - B_t^{m_0}) \\
&\le \frac{\kappa}{N}\sum_{k=1}^N \Big[-\psi_t^{kM_0}\Big(R_\infty(\theta_t^{M_0 k} + \delta) - \delta\Big) - \psi_t^{km_0}\Big(R_\infty(\theta_t^{km_0} + \delta) - \delta\Big)\Big]dt \\
&\quad + \sqrt{2\sigma}d(B_t^{M_0} - B_t^{m_0}) \\
&\le \frac{\kappa}{N}\sum_{k=1}^N \Big[2\delta\psi_M - \psi_m R_\infty(\theta_t^{M_0 m_0} + 2\delta)\Big]dt + \sqrt{2\sigma}d(B_t^{M_0} - B_t^{m_0}) \\
&= \kappa[2\delta\psi_M - \psi_m R_\infty(\theta_t^{M_0 m_0} + 2\delta)]dt + \sqrt{2\sigma}d(B_t^{M_0} - B_t^{m_0}).
\end{aligned} \tag{3.3.24}$$

Here in the fourth equality, we used the relations:

$$\begin{aligned} &R_\infty(\theta_t^{M_0 k} + \delta) = \frac{\sin D_\infty}{D_\infty}(\theta_t^{M_0 k} + \delta) \le \sin(\theta_t^{M_0 k} + \delta), \\ &\text{and} \quad R_\infty(\theta_t^{k m_0} + \delta) \le \sin(\theta_t^{k m_0} + \delta). \end{aligned}$$

Thus, relation (3.3.24) can be rewritten as

$$d(\theta_t^{M_0 m_0} + 2\delta) \le \kappa[2\delta\psi_M - \psi_m R_\infty(\theta_t^{M_0 m_0} + 2\delta)]dt + \sqrt{2\sigma} d(B_t^{M_0} - B_t^{m_0}). \tag{3.3.25}$$

We apply Ito's lemma to $(\theta_t^{M_0 m_0} + 2\delta)e^{\kappa\psi_m R_\infty t}$ using (3.3.25) to see

$$d\Big((\theta_t^{M_0 m_0} + 2\delta)e^{\kappa\psi_m R_\infty t}\Big) \le 2\kappa\delta\psi_M e^{\kappa\psi_m R_\infty t}dt + \sqrt{2\sigma}e^{\kappa\psi_m R_\infty t} d(B_t^{M_0} - B_t^{m_0}).$$

This yields that for $t \le \tau^0(\delta) \wedge \tau^0_{D_\infty}$,

$$\Big(\theta_t^{M_0 m_0} + 2\delta\Big) \le \Big(\theta_0^{M_0 m_0} + 2\delta\Big)e^{-\kappa\psi_m R_\infty t} + \frac{2\delta\psi_M}{\psi_m R_\infty}(1 - e^{-\kappa\psi_m R_\infty t}) + \sqrt{2\sigma}\tilde{Z}_t^{(0)},$$

where the O-U process $\tilde{Z}_t^{(0)}$ is given by the following relation:

$$\tilde{Z}_t^{(0)} := \int_0^t e^{-\kappa\psi_m R_\infty (t-s)} d(B_s^{M_0} - B_s^{m_0}).$$

Next, we also define the zeroth barrier function and stopping times:

$$\begin{aligned} L_0(s) &:= (D(\Theta_0) + 2\delta)e^{-\kappa\psi_m R_\infty s} + \frac{2\delta\psi_M}{\psi_m R_\infty}(1 - e^{-\kappa\psi_m R_\infty s}) + 2\delta, \\ \tau^*_{L_0} &:= \inf\left\{s > 0 : \theta_s^{M_0 m_0} + 2\delta \ge L_0(s)\right\}, \\ \tau_{L_0} &:= \inf\left\{s > 0 : D(\Theta_s) \ge L_0(s)\right\}. \end{aligned} \tag{3.3.26}$$

Below, we provide some stochastic estimates for $\tilde{Z}_t^{(0)}$ and τ_{L_0}. For notational simplicity, we set

$$P_1 := A\left(\frac{\delta}{\sqrt{2\sigma}}, \kappa\psi_m R_\infty, T_\delta\right) e^{-\frac{\kappa\psi_m R_\infty \delta^2}{4\sigma}}, \qquad P_2 := 16N\sqrt{\frac{\sigma}{\pi\delta\kappa C_\delta}} e^{-\frac{\delta\kappa C_\delta}{16\sigma}}.$$

Lemma 3.3.3. *Suppose that the coupling strength κ is sufficiently large such that*

$$r\left(\delta\sqrt{\frac{\kappa\psi_m R_\infty}{\sigma}}, \kappa\psi_m R_\infty T_\delta\right) \le r_0.$$

Then, we have

$$(i)\ \mathbb{P}\left\{\sup_{0\le t\le T_\delta}\sqrt{2\sigma}\tilde{Z}_t^{(0)} > 2\delta\right\} \le P_1,$$
$$(ii)\ \mathbb{P}\left\{\tau_{L_0} < T_\delta\right\} \le P_1 + P_2 =: P_\infty.$$

Proof. (i) We choose κ sufficiently large such that

$$r\left(\delta\sqrt{\frac{\kappa\psi_m R_\infty}{\sigma}}, \kappa\psi_m R_\infty T_\delta\right) \le r_0.$$

Then, we have

$$\mathbb{P}\left\{\sup_{0\le t\le T_\delta}\sqrt{2\sigma}\tilde{Z}_t^{(0)} > 2\delta\right\} \le A\left(\frac{\delta}{\sqrt{2\sigma}}, \kappa\psi_m R_\infty, T_\delta\right)e^{-\frac{\kappa\psi_m R_\infty\delta^2}{4\sigma}} = P_1.$$

(ii) Note that the relations

$$D(\Theta_0) = \theta_0^{M_0 m_0} \quad \text{and} \quad D(\Theta_t) \le \theta_t^{M_0 m_0} + 2\delta \quad \text{for } t \le \tau^0(\delta)$$

imply

$$\begin{aligned}&\mathbb{P}\left\{\exists s \le T_\delta \wedge \tau^0(\delta) \wedge \tau^0_{D_\infty} : \theta_s^{M_0 m_0} + 2\delta \ge L_0(s)\right\}\\ &\qquad\qquad \le \mathbb{P}\left\{\sup_{0\le t\le T}\sqrt{2\sigma}\tilde{Z}_t^{(0)} \ge 2\delta\right\} \le P_1.\end{aligned} \tag{3.3.27}$$

Then, we bound the probability for the event $\{\tau_{L_0} < T_\delta\}$ as follows.

$$\begin{aligned}&\mathbb{P}\left\{\tau_{L_0} < T_\delta\right\} \le \mathbb{P}\left\{\tau_{L_0}\wedge\tau^0(\delta) < T_\delta\right\} \le \mathbb{P}\left\{\tau^*_{L_0}\wedge\tau^0(\delta) < T_\delta\right\}\\ &\le \mathbb{P}\left\{\tau^*_{L_0} \le T_\delta\wedge\tau^0(\delta)\wedge\tau^0_{D_\infty}\right\} + \mathbb{P}\left\{\tau^0(\delta) < T_\delta\right\}\\ &\qquad\qquad + \mathbb{P}\left\{\tau^0_{D_\infty} \le T_\delta\wedge\tau^0(\delta)\wedge\tau^*_{L_0}\right\}\\ &=: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}.\end{aligned} \tag{3.3.28}$$

Here we used the following relations:

1. $\tau_{L_0}\wedge\tau^0(\delta) \le \tau_{L_0}$: This is clear.

2. $\tau^*_{L_0} \wedge \tau^0(\delta) \leq \tau_{L_0} \wedge \tau^0(\delta)$: This is clear when $\tau_{L_0} \geq \tau^0(\delta)$.
   If $\tau_{L_0} < \tau^0(\delta)$, the set
   $$\left\{t > 0 : D(\Theta_t) \geq L_0(t) \text{ and } t \leq \tau^0(\delta)\right\}$$
   is nonempty and contained in
   $$\left\{t > 0 : \theta_t^{M_0 m_0} + 2\delta \geq L_0(t) \text{ and } t \leq \tau^0(\delta)\right\},$$
   since $D(\Theta_t) \leq \theta_t^{M_0 m_0} + 2\delta$ for $t \leq \tau^0(\delta)$.
   Therefore, we have
   $$\begin{aligned}\tau^*_{L_0} &= \inf\left\{t > 0 : \theta_t^{M_0 m_0} + 2\delta \geq L_0(t) \text{ and } t \leq \tau^0(\delta)\right\}\\ &\leq \inf\left\{t > 0 : D(\Theta_t) \geq L_0(t) \text{ and } t \leq \tau^0(\delta)\right\} = \tau_{L_0},\end{aligned}$$
   and
   $$\tau^*_{L_0} \wedge \tau^0(\delta) = \tau^*_{L_0} \leq \tau_{L_0} = \tau_{L_0} \wedge \tau^0(\delta).$$

3. The last inequality comes from the following relations:
   $$\begin{aligned}&\left\{\tau^*_{L_0} \wedge \tau^0(\delta) < T_\delta\right\} - \left\{\tau^0(\delta) < T_\delta\right\}\\ &= \left\{\tau^*_{L_0} < T_\delta \leq \tau^0(\delta)\right\} \subset \left\{\tau^*_{L_0} < T_\delta \wedge \tau^0(\delta)\right\}\\ &\subset \Big(\left\{\tau^*_{L_0} \leq T_\delta \wedge \tau^0(\delta) \wedge \tau^0_{D_\infty}\right\} \cup \left\{\tau^0_{D_\infty} \leq T_\delta \wedge \tau^0(\delta) \wedge \tau^*_{L_0}\right\}\Big).\end{aligned}$$

Below, we estimate the terms $\mathcal{I}_{1i},\ i = 1, 2, 3$ separately.

⋄ (Estimate of $\mathcal{I}_{11}$): We use (3.3.27) to get
$$\mathcal{I}_{11} \leq P_1.$$

⋄ (Estimate of $\mathcal{I}_{12}$): We use Lemma 3.3.2 to obtain
$$\mathcal{I}_{12} \leq 16N\sqrt{\frac{\sigma}{\pi\delta\kappa C_\delta}}e^{-\frac{\delta\kappa C_\delta}{16\sigma}} =: P_2.$$

⋄ (Estimate of $\mathcal{I}_{13}$): It follows from $(3.3.18)_1$ and (3.3.26) that
$$\begin{aligned}L_0(s) &= (D(\Theta_0) + 2\delta)e^{-\kappa\psi_m R_\infty s} + \frac{2\delta\psi_M}{\psi_m R_\infty}(1 - e^{-\kappa\psi_m R_\infty s}) + 2\delta\\ &\leq \max\left\{D(\Theta_0) + 2\delta, \frac{2\delta\psi_M}{\psi_m R_\infty}\right\} + 2\delta < D_\infty, \qquad \forall s \leq T_\delta.\end{aligned}$$

This implies

$$\mathcal{I}_{13} = 0.$$

Finally, in (3.3.28), we combine all the estimates $\mathcal{I}_{1i},\ i = 1, 2, 3$ to derive the desired estimate. □

Step B (iterative time-zone estimates)

In this step, we consider the time interval $[(\ell - 1)T_\delta, \ell T_\delta]$ to see how the iterative estimate works. First, we set $\ell = 2$. If we assume that τ_{L_0} is larger than or equal to T_δ, then at the instant T_δ, we have

$$D(\Theta_{T_\delta}) \le L_0(T_\delta) = (D(\Theta_0)+2\delta)e^{-\kappa\psi_m R_\infty T_\delta}+\frac{2\delta\psi_M}{\psi_m R_\infty}(1-e^{-\kappa\psi_m R_\infty T_\delta})+2\delta < D_\infty.$$

From the data at T_δ, we can define new indices M_1 and m_1 satisfying

$$D(\Theta_{T_\delta}) := \theta^{M_1}_{T_\delta} - \theta^{m_1}_{T_\delta}.$$

We define the first barrier function $L_1(s)$ as follows:

$$\begin{aligned} &L_1(s)\\ &:= L_0(s)\chi_{(0,T_\delta]}\\ &\quad + \left[(L_0(T_\delta) + 2\delta)e^{-\kappa\psi_m R_\infty(s-T_\delta)} + \frac{2\delta\psi_M}{\psi_m R_\infty}(1 - e^{-\kappa\psi_m R_\infty(s-T_\delta)}) + 2\delta\right]\chi_{(T_\delta,\infty)}\\ &= L_0(s) + \chi_{(T_\delta,\infty)}\cdot 4\delta e^{-\kappa\psi_m R_\infty(s-T_\delta)}, \end{aligned}$$

where we used the defining relation of $L_0(T_\delta)$ in the second identity.

Next, we define four new stopping times analogous to (3.3.1) - (3.3.4):

$$\begin{aligned} \tau^1(\delta) &:= \inf\left\{s > T_\delta : \theta^i_s \notin (\theta^{m_1}_s - \delta, \theta^{M_1}_s + \delta) \quad \text{for some } i\right\},\\ \tau^1_{D_\infty} &:= \inf\left\{s > T_\delta : \theta^{M_1 m_1}_s + 2\delta \ge D_\infty\right\},\\ \tau^*_{L_1} &:= \inf\left\{s > T_\delta : \theta^{M_1 m_1}_s + 2\delta \ge L_1(s)\right\},\\ \tau_{L_1} &:= \inf\left\{s > 0 : D(\Theta_s) \ge L_1(s)\right\}. \end{aligned}$$

For $T_\delta \le t \le \tau^1(\delta) \wedge \tau^1_{D_\infty}$, we can formulate the copy of (3.3.24):

$$d(\theta^{M_1 m_1}_t + 2\delta) \le \kappa[2\delta\psi_M - \psi_m R_\infty(\theta^{M_1 m_1}_t + 2\delta)]dt + \sqrt{2\sigma}d(B^{M_1}_t - B^{m_1}_t).$$

By the same argument as in Section 3.3.5, we have

$$\theta_t^{M_1m_1}+2\delta \leq (\theta_{T_\delta}^{M_1m_1}+2\delta)e^{-\kappa\psi_m R_\infty(t-T_\delta)}+\frac{2\delta\psi_M}{\psi_m R_\infty}\left(1-e^{-\kappa\psi_m R_\infty(t-T_\delta)}\right)+\sqrt{2\sigma}\tilde{Z}_t^{(1)},$$

where the O-U process $\tilde{Z}_t^{(1)}$ is given by

$$\tilde{Z}_t^{(1)} := \int_0^{t-T_\delta} e^{-\kappa\psi_m R_\infty(t-T_\delta-s)}d(B_{s+T_\delta}^{M_1} - B_{s+T_\delta}^{m_1}).$$

Similar to Lemma 3.3.3, we have the following lemma.

Lemma 3.3.4. *The following estimates hold.*

(i) $\mathbb{P}\left\{\sup_{T_\delta\leq s\leq 2T_\delta} \sqrt{2\sigma}\tilde{Z}_s^{(1)} \geq 2\delta\right\} \leq P_1.$

(ii) $\mathbb{P}\left\{\exists s : T_\delta < s \leq 2T_\delta \wedge \tau^1(\delta) \wedge \tau^1_{D_\infty},\ \theta_s^{M_1m_1} + 2\delta \geq L_1(s)\middle|\tau_{L_0} \geq T_\delta\right\} \leq P_1.$

Proof. (i) The first estimate can be done as in Lemma 3.3.3:

$$\mathbb{P}\left\{\sup_{T_\delta\leq t\leq 2T_\delta} \sqrt{2\sigma}\tilde{Z}_t^{(1)} \geq 2\delta\right\} \leq A\left(\frac{\delta}{\sqrt{2\sigma}}, \kappa\psi_m R_\infty, T_\delta\right)e^{-\frac{\kappa\psi_m R_\infty\delta^2}{4\sigma}} = P_1.$$

(ii) We use a similar argument to (3.3.27) and (i) to obtain

$$\begin{aligned}\mathbb{P}&\left\{\exists s : T_\delta < s \leq 2T_\delta \wedge \tau^1(\delta) \wedge \tau^1_{D_\infty},\ \theta_s^{M_1m_1} + 2\delta \geq L_1(s)\middle|\tau_{L_0} \geq T_\delta\right\}\\ &\leq \mathbb{P}\left\{\sup_{T_\delta\leq s\leq 2T_\delta} \sqrt{2\sigma}\tilde{Z}_s^{(1)} \geq 2\delta\right\} \leq P_1.\end{aligned} \tag{3.3.29}$$

□

Therefore, we get the required estimations for $[T_\delta, 2T_\delta]$. For the general step $[(\ell-1)T_\delta, \ell T_\delta]$, we may proceed similar estimates by considering desired data at the time $(\ell-1)T_\delta$.

Suppose that $\ell T_\delta \leq \tau_{L_{\ell-1}}$, and set indices M_ℓ and m_ℓ such that

$$D(\Theta_{\ell T_\delta}) = \theta_{\ell T_\delta}^{M_\ell} - \theta_{\ell T_\delta}^{m_\ell},$$

and introduce ℓ-th barrier function L_ℓ and ℓ-th stopping times as follows:

$$\begin{aligned}
\tau^\ell_{D_\infty} &:= \inf\left\{s > \ell T_\delta : \theta^{M_\ell m_\ell}_s + 2\delta > D_\infty\right\},\\
\tau^\ell(\delta) &:= \inf\left\{s > \ell T_\delta : \theta^i_s \notin \left(\theta^{m_\ell}_s - \delta,\ \theta^{M_\ell}_s + \delta\right) \text{ for some } i\right\},\\
L_\ell(s) &:= L_{\ell-1}(s) + 4\delta e^{-\kappa\psi_m R_\infty(s-\ell T_\delta)}\chi_{(\ell T_\delta,\infty)}\\
&\quad = L_0(s) + 4\delta\sum_{r=1}^{\ell} e^{-\kappa\psi_m R_\infty(s-rT_\delta)}\chi_{(rT_\delta,\infty)},\\
\tau^*_{L_\ell} &:= \inf\left\{s > \ell T_\delta : \theta^{M_\ell m_\ell}_s + 2\delta > L_\ell(s)\right\},\\
\tau_{L_\ell} &:= \inf\left\{s > 0 : D(\Theta_s) > L_\ell(s)\right\}.
\end{aligned}$$

For $\ell T_\delta \le t \le \tau^\ell(\delta) \wedge \tau^\ell_{D_\infty}$, we have

$$\theta^{M_\ell m_\ell}_t + 2\delta \le (\theta^{M_\ell m_\ell}_{\ell T_\delta} + 2\delta)e^{-\kappa\psi_m R_\infty(t-\ell T_\delta)} + \frac{2\delta\psi_M}{\psi_m R_\infty}(1 - e^{-\kappa\psi_m R_\infty(t-\ell T_\delta)}) + \sqrt{2\sigma}\tilde{Z}^{(\ell)}_t,$$

where the O-U process $\tilde{Z}^{(\ell)}_t$ is given as

$$\tilde{Z}^{(\ell)}_t := \int_0^{t-\ell T_\delta} e^{-2\kappa\psi_m R_\infty(t-\ell T_\delta - s)} d(B^{M_\ell}_{s+\ell T_\delta} - B^{m_\ell}_{s+\ell T_\delta}).$$

Then, we have

$$\begin{aligned}
&\mathbb{P}\left\{\exists s : \ell T_\delta < s \le (\ell+1)T_\delta \wedge \tau^\ell(\delta) \wedge \tau^\ell_{D_\infty},\ \theta^{M_\ell m_\ell}_s + 2\delta \ge L_\ell(s) \,\middle|\, \tau_{L_{\ell-1}} \ge \ell T_\delta\right\}\\
&\qquad = \mathbb{P}\left\{\tau^*_{L_\ell} < (\ell+1)T_\delta \wedge \tau^\ell(\delta) \wedge \tau^\ell_{D_\infty} \,\middle|\, \tau_{L_{\ell-1}} \ge \ell T_\delta\right\},\\
&\mathbb{P}\left\{\tau^*_{L_\ell} < (\ell+1)T_\delta \wedge \tau^\ell(\delta) \wedge \tau^\ell_{D_\infty} \,\middle|\, \tau_{L_{\ell-1}} \ge \ell T_\delta\right\}\\
&\qquad \le \mathbb{P}\left\{\sup_{\ell T_\delta \le t \le (\ell+1)T_\delta} \sqrt{2\sigma}\tilde{Z}^{(\ell)}_t > 2\delta\right\} \le P_1.
\end{aligned}$$

Step C (Derivation of the recursive inequality)

Now, we are ready to derive the recursive relation (3.3.20). First, we define a global barrier function and stopping time:

$$\begin{aligned}
L(s) &:= L_0(s) + 2\delta\sum_{r=1}^{\infty} e^{-\kappa\psi_m R_\infty(s-rT_\delta)}\chi_{(rT_\delta,\infty)},\\
\tau_L &:= \inf\left\{s > 0 : D(\Theta_s) > L(s)\right\}.
\end{aligned}$$

Note that, for any integer ℓ, we have

$$L(s) = L_\ell(s) \quad \text{for } 0 \le s \le (\ell+1)T_\delta,$$

so that we have the equivalence between events:

$$\{\tau_L < (\ell+1)T_\delta\} \iff \{\tau_{L_\ell} < (\ell+1)T_\delta\}. \tag{3.3.30}$$

Therefore, we get

$$\begin{aligned}
&\mathbb{P}\{\tau_L < (\ell+1)T_\delta\} \\
&\quad = \mathbb{P}\{\tau_L < \ell T_\delta\} + \mathbb{P}\{\ell T_\delta \le \tau_L < (\ell+1)T_\delta\} \\
&\quad = \mathbb{P}\{\tau_L < \ell T_\delta\} + \underbrace{\mathbb{P}\left\{\tau_L < (\ell+1)T_\delta \middle| \tau_L \ge \ell T_\delta\right\}}_{=:\Delta} \mathbb{P}\{\tau_L \ge \ell T_\delta\}.
\end{aligned} \tag{3.3.31}$$

In the above equality, the conditional probability Δ can be bounded using (3.3.30) as follows.

$$\begin{aligned}
\Delta &= \mathbb{P}\left\{\tau_{L_\ell} < (\ell+1)T_\delta \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} \\
&\le \mathbb{P}\left\{\tau_{L_\ell} \wedge \tau^\ell(\delta) < (\ell+1)T_\delta \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} \\
&\le \mathbb{P}\left\{\tau^*_{L_\ell} \wedge \tau^\ell(\delta) < (\ell+1)T_\delta \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} \\
&\le \mathbb{P}\left\{\tau^*_{L_\ell} \le (\ell+1)T_\delta \wedge \tau^\ell(\delta) \wedge \tau^\ell_{D_\infty} \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} \\
&\quad + \mathbb{P}\left\{\tau^\ell(\delta) < (\ell+1)T_\delta \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} \\
&\quad + \mathbb{P}\left\{\tau^\ell_{D_\infty} \le (\ell+1)T_\delta \wedge \tau^*_{L_\ell} \wedge \tau^\ell(\delta) \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} \\
&=: \mathcal{I}_{21} + \mathcal{I}_{22} + \mathcal{I}_{23}.
\end{aligned} \tag{3.3.32}$$

Next, we estimate the terms $\mathcal{I}_{2i}$, $i = 1, 2, 3$, one by one.

$\diamond$ (Estimate of $\mathcal{I}_{2i},\ i = 1, 2$): We use similar arguments as in (3.3.29) or Lemma 3.3.3 to get

$$\mathcal{I}_{21} \le P_1, \quad \mathcal{I}_{22} \le P_2. \tag{3.3.33}$$

$\diamond$ (Estimate of $\mathcal{I}_{23}$): For $\mathcal{I}_{23}$, we use the condition (3) in (3.3.18) to deduce

$$\mathcal{I}_{23} = 0,$$

where we used the relation:

$$L_\ell(s) \le L(s) \le \max\left\{ D(\Theta_0) + 2\delta,\ \frac{2\delta\psi_M}{\psi_m R_\infty} + 2\delta + \frac{4\delta}{1 - e^{-\frac{\psi_m R_\infty \delta}{2C_\delta}}} \right\} < D_\infty.$$

Finally, we combine (3.3.31)–(3.3.33) to obtain

$$\begin{aligned}
\mathbb{P}\left\{\tau_L < (\ell+1)T_\delta\right\} &= \mathbb{P}\left\{\tau_L < \ell T_\delta\right\} \\
&\qquad + \mathbb{P}\left\{\tau_L < (\ell+1)T_\delta \,\middle|\, \tau_L \ge \ell T_\delta\right\} \mathbb{P}\left\{\tau_L \ge \ell T_\delta\right\} \\
&\le \mathbb{P}\left\{\tau_L < \ell T_\delta\right\} + P_\infty \mathbb{P}\left\{\tau_L \ge \ell T_\delta\right\},
\end{aligned} \tag{3.3.34}$$

or equivalently, we get (3.3.21):

$$\mathbb{P}\left\{\tau_L \ge (\ell+1)T_\delta\right\} \ge (1 - P_\infty)\mathbb{P}\left\{\tau_L \ge \ell T_\delta\right\}.$$

From the induction on ℓ and Lemma 3.3.3, we conclude Theorem 3.3.1:

$$\mathbb{P}\left\{\tau_L \ge \ell T_\delta\right\} \ge (1 - P_\infty)^{\ell-1}\mathbb{P}\left\{\tau_L \ge T_\delta\right\} \ge (1 - P_\infty)^{\ell}.$$

Remark 3.3.3. *For completeness, we need to check whether the condition* (3.3.18) *can be achievable. If we take a limit $\delta \to 0$, the left-hand side of condition* (3.3.18)*(i) becomes*

$$\max\left\{ D(\Theta_0), \frac{8D_\infty(\psi_M - \psi_m)}{\psi_m \sin D_\infty} \right\}.$$

Therefore, if $D(\Theta_0) < D_\infty$ and $\frac{\psi_M}{\psi_m} < 1 + \frac{\sin D_\infty}{8}$, there exists a positive δ satisfying condition (3.3.18)*(i). In addition, the probability estimate P_∞ also gives meaningful values (that is, $P_\infty < 1$) only if $\frac{\delta}{\sqrt{2\sigma}}$ and κ are sufficiently large for fixed T_δ.*

3.4 Order parameter estimate

We study a behavior of the expectation of order parameter square $R^2(\Theta)$ when ψ is a perturbation of constant function $\psi \equiv 1$.

Lemma 3.4.1. *Let $(X_t, \Theta_t)_{t\geq 0}$ be a solution process of* (3.0.1)*. Then, we have the following formula for $R^2(\Theta_t)$:*

$$dR_t^2 = \left(\frac{2\sigma}{N} - 2\sigma R_t^2 + \frac{2\kappa}{N^3} \sum_{i,j,k=1}^{N} \psi(\|x_i - x_k\|) \sin(\theta_t^k - \theta_t^i) \sin(\theta_t^j - \theta_t^i) \right) dt \\ - \frac{\sqrt{2\sigma}}{N^2} \sum_{i,j=1}^{N} \sin(\theta_t^i - \theta_t^j)(dB_t^i - dB_t^j).$$

Proof. We use Proposition 3.1.2 and Itô's formula to obtain

$$\begin{aligned} dR_t^2 &= \frac{1}{N^2} \sum_{i\neq j} d(\cos(\theta_t^i - \theta_t^j)) \\ &= \frac{1}{N^2} \sum_{i\neq j} \left(-\sin(\theta_t^i - \theta_t^j) d(\theta_t^i - \theta_t^j) - \frac{1}{2} \cos(\theta_t^i - \theta_t^j) \left(d(\theta_t^i - \theta_t^j) \right)^2 \right) \\ &= \frac{1}{N^2} \sum_{i\neq j} \left(-\sin(\theta_t^i - \theta_t^j) d(\theta_t^i - \theta_t^j) - 2\sigma \cos(\theta_t^i - \theta_t^j) dt \right) \\ &= \frac{2\sigma}{N} dt + \frac{1}{N^2} \sum_{i,j=1}^{N} \left(-\sin(\theta_t^i - \theta_t^j) d(\theta_t^i - \theta_t^j) - 2\sigma \cos(\theta_t^i - \theta_t^j) dt \right) \\ &= \left(\frac{2\sigma}{N} - 2\sigma R_t^2 \right) dt + \frac{1}{N^2} \sum_{i,j=1}^{N} \left(-\sin(\theta_t^i - \theta_t^j) d(\theta_t^i - \theta_t^j) \right) \\ &= \left(\frac{2\sigma}{N} - 2\sigma R_t^2 + \frac{2\kappa}{N^3} \sum_{i,j,k=1}^{N} \psi(\|x_i - x_k\|) \sin(\theta_t^k - \theta_t^i) \sin(\theta_t^j - \theta_t^i) \right) dt \\ &\qquad - \frac{\sqrt{2\sigma}}{N^2} \sum_{i,j=1}^{N} \sin(\theta_t^i - \theta_t^j)(dB_t^i - dB_t^j). \end{aligned}$$

□

Next, we present a preparatory lemma to apply a comparison principle for dR^2. From the following lemma, we can find an upper bound of the drift term of dR^2 in terms of R^2 when the communication ψ is assumed to be a constant function.

Lemma 3.4.2. *Let $\Theta = (\theta^1, \cdots, \theta^N)$ be any angle configuration. Then, we have*

$$\frac{1}{N^3}\sum_{i,j,k=1}^{N}\sin(\theta^k-\theta^i)\sin(\theta^j-\theta^i) \le R(\Theta)^2(1-R(\Theta)^2). \tag{3.4.1}$$

Proof. First, from the definition of order parameter in (3.1.5), we have

$$\begin{aligned}
R(\Theta)^2 &= \left|\frac{1}{N}\sum_{k=1}^{N}e^{\mathrm{i}\theta^k}\right|^2 = \left|\frac{1}{N}\sum_{k=1}^{N}e^{\mathrm{i}\theta^k-\mathrm{i}\theta^j}\right|^2 \\
&= \left(\frac{1}{N}\sum_{k=1}^{N}\cos(\theta^k-\theta^j)\right)^2 + \left(\frac{1}{N}\sum_{k=1}^{N}\sin(\theta^k-\theta^j)\right)^2, \quad \forall\, j=1,\cdots,N.
\end{aligned}$$

Therefore, as we subtract the left-hand side of (3.4.1) from the right-hand side, we have

$$\begin{aligned}
R.H.S - L.H.S &= R^2 - R^4 - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{k=1}^{N}\sin(\theta^k-\theta^i)\right)^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{k=1}^{N}\cos(\theta^k-\theta^i)\right)^2 - R^4.
\end{aligned}$$

Finally, we can deduce the desired result by using Cauchy-Schwarz inequality and Proposition 3.1.2:

$$\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{k=1}^{N}\cos(\theta^k-\theta^i)\right)^2 \ge \left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{N}\sum_{k=1}^{N}\cos(\theta^k-\theta^i)\right)^2 = R^4.$$

□

Now, we are ready to verify the instability of heading angle alignment state for (3.0.1) in terms of the order parameter.

Theorem 3.4.1. *Let $(X_t, \Theta_t)_{t\geq 0}$ be a solution process of* (3.0.1), *and assume that there exists a positive constant $\varepsilon < \frac{(N-1)\sigma}{N\kappa}$ satisfying*

$$|\psi(x) - 1| \leq \varepsilon, \quad \forall\, x \in \mathbb{R}.$$

Then, we have

$$\limsup_{t\to\infty} \mathbb{E}[R_t^2] \leq \frac{1 - a + \sqrt{(1-a)^2 + \frac{4a}{N} + 4\varepsilon}}{2} < 1, \quad a := \frac{\sigma}{\kappa}.$$

Proof. From Lemma 3.4.1 and Lemma 3.4.2, one can obtain the following inequality:

$$\begin{aligned}
\frac{d\mathbb{E}[R_t^2]}{dt} &\leq \frac{2\sigma}{N} + 2\kappa\varepsilon - 2\sigma\mathbb{E}[R_t^2] + 2\kappa\mathbb{E}[R_t^2 - R_t^4] \\
&\leq \frac{2\sigma}{N} + 2\kappa\varepsilon - 2\sigma\mathbb{E}[R_t^2] + 2\kappa(\mathbb{E}[R_t^2] - \mathbb{E}[R_t^2]^2) \\
&= -2\kappa\mathbb{E}[R_t^2]^2 + (2\kappa - 2\sigma)\mathbb{E}[R_t^2] + \frac{2\sigma}{N} + 2\kappa\varepsilon.
\end{aligned} \tag{3.4.2}$$

Now, suppose on the contrary that

$$\limsup_{t\to\infty} \mathbb{E}[R_t^2] =: c_0 > \frac{1 - a + \sqrt{(1-a)^2 + \frac{4a}{N} + 4\varepsilon}}{2}. \tag{3.4.3}$$

Then, for any real number c_1 between c_0 and $\frac{1-a+\sqrt{(1-a)^2+\frac{4a}{N}+4\varepsilon}}{2}$, there exists a time $t_0 \geq 0$ such that $\mathbb{E}[R_t^2] > c_1$ for any $t > t_0$. However, we have a negative upper bound on $\frac{d\mathbb{E}[R_t^2]}{dt}$ by using (3.4.2):

$$\frac{d\mathbb{E}[R_t^2]}{dt} \leq -2\kappa c_1^2 + (2\kappa - 2\sigma)c_1 + \frac{2\sigma}{N} + 2\kappa\varepsilon < 0, \quad \forall\, t > t_0,$$

and this contradicts to the assumption (3.4.3). Therefore, we conclude the desired upper bound on $\limsup_{t\to\infty} \mathbb{E}[R_t^2]$.

□

Remark 3.4.1. *According to the analysis, one can see that the dependency of ψ on x is not really important. In fact, even if ψ is independent to the dynamics, Theorem 3.4.1 still holds if ψ is a small perturbation of constant function* 1.

3.5 Numerical simulations

We perform a numerical simulations for (3.0.1) and verify the temporal evolution of order parameters. For all simulations, we used the Milstein method for SDE under the following system parameters:

$$\kappa = 0.1, \quad \sigma = 0.01, \quad N = 100, \quad \Delta t = 0.01.$$

Then, we first plot R^2 versus t graphs for $\psi \equiv 1$ and ψ perturbed from 1 for one sample path. Here, we just used a random perturbation from $\psi \equiv 1$ via uniform distribution.

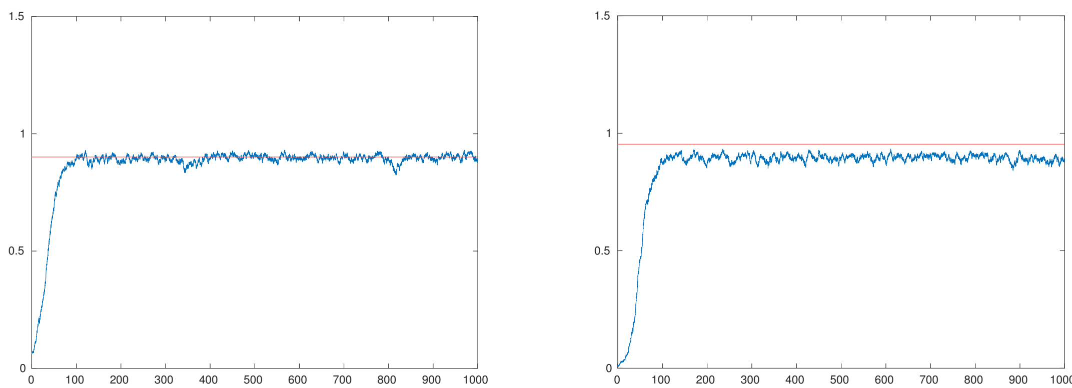


Figure 3.1: Temporal evolutions of R^2 for $\varepsilon = 0$ and $\varepsilon = 0.05$.

In each figure in Fig 3.1, the red horizontal line shows the upper bound of $\mathbb{E}[R^2]$ in Theorem 3.4.1. If ψ is defined as the constant function 1, i.e., the Kuramoto model with additive noise, R^2 vibrates irregularly near the predetermined upper bound of its expectation, although it does not really 'converge' to the red line because of the additive noise. On the other hand, if ψ is perturbed from 1, then the sample path vibrates at the below of the red reference line. Since we removed the x-dependency from (3.0.1) for these figures, the evolution of R^2 might be slightly different from the above figure when ψ is a nonconstant function close to 1 uniformly. Here, as the distribution of perturbed ψ is centered at 1, the temporal evolution of R^2 shows similar long-time behavior from the non-perturbed one.

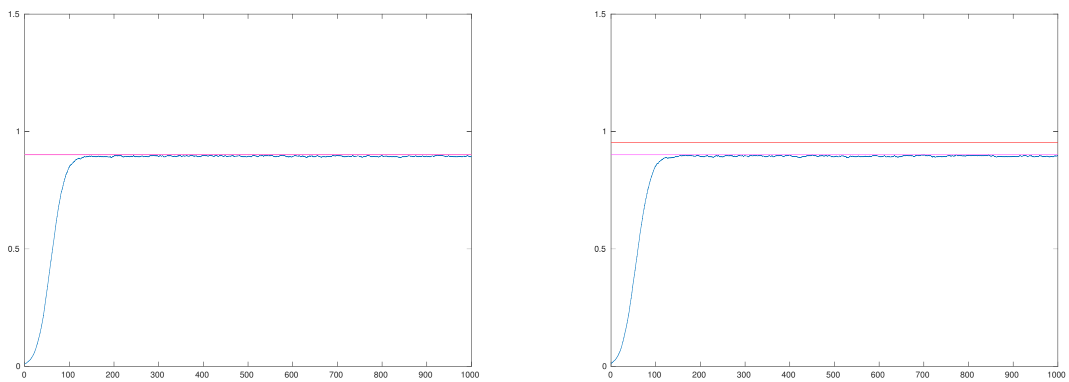


Figure 3.2: Temporal evolutions of $\mathbb{E}[R^2]$ for $\varepsilon = 0$ and $\varepsilon = 0.05$.

Fig. 3.2 shows the empirical mean of order parameter squares for 50 sample paths under the same setting with Fig 3.1. We here chose 50 different initial data from uniform distribution on $\mathbb{S}^1$, but the average of the order parameter squares converges to the suggested upper bound (magenta horizontal line) for $\psi \equiv 1$ in both cases.

Chapter 4

J-K model with multiplicative noises

We now study a heading angle alignment of the stochastic J-K model (1.0.8) under a multiplicative noise. To be precise, we are interested in the random perturbation of the communication weight ψ, which is represented by a multiplicative noise.

We here present sufficient conditions leading to the heading angle alignment in terms of the system parameters and initial data. Our analysis begins with the two-body system, where it turns out to be stable under the effect of noise, showing that the heading angle alignment occurs when the communication is sufficiently strong with respect to the noise strength. For the general many-body system with a corresponding condition, we show the accumulation of heading angles modulo π and the stochastic stability of heading angle alignment under the assumption of the constant communication weight, which suggests a strong evidence for the heading angle alignment. This analysis is done by transporting the system into a similar form to the stochastic Kuramoto model, where we refined the order parameter analysis in order to extend local stochastic stability results to the whole circle of heading angles. We also provide several numerical simulations and compare them with analytical results. We note that this chapter is based on the joint work [51].

4.1 Basic properties

We briefly review some basic properties of the stochastic J-K models with multiplicative noise. We also present several a priori estimates for later use.

For $\Theta = (\theta^1, \cdots, \theta^N)$ and $n \in \mathbb{N}$, we introduce the order parameters (R_n, ϕ_n) for modulo $(2\pi/n)$ alignment by the following implicit relation:

$$R_n(\Theta)e^{\mathrm{i}n\phi_n(\Theta)} := \frac{1}{N}\sum_{k=1}^{N} e^{\mathrm{i}n\theta^k}, \quad R_n(\Theta) > 0 \quad \text{and} \quad \phi_n(\Theta) \in \mathbb{R}/\left(\frac{2\pi}{n}\mathbb{Z}\right). \tag{4.1.1}$$

In particular, if there is no confusion, we simply write (R_1, ϕ_1) as (R, ϕ).

4.1.1 Derivation of multiplicative noise J-K model

We here consider a modeling of the multiplicative communication weight in the J-K model. First, we formally write the communication weight

$$\kappa\psi(\|x_t^k - x_t^j\|)$$

acting from k to j as the deterministic part $\kappa\widetilde{\psi}(\|x_t^k - x_t^j\|)$ and a white noise part $\sqrt{2\sigma}\dot{B}_t^j$, i.e.,

$$\kappa\psi(\|x_t^k - x_t^j\|) = \kappa\tilde{\psi}(\|x_t^k - x_t^j\|) + \sqrt{2\sigma}\dot{B}_t^j, \quad \forall\ k = 1, \cdots, N. \tag{4.1.2}$$

Denoting $\widetilde{\psi}$ as ψ for simplicity, we have the multiplicative noise J-K model (1.0.8) from (1.0.4) and (4.1.2):

$$\begin{cases} dx_t^j = (\cos\theta_t^j, \sin\theta_t^j)\ dt, \quad t > 0,\ j = 1, \cdots, N, \\ d\theta_t^j = \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^{N} \psi(\|x_t^k - x_t^j\|)\sin(\theta_t^k - \theta_t^j)dt + \dfrac{\sqrt{2\sigma}}{N}\sum_{k=1}^{N}\sin(\theta_t^k - \theta_t^j)dB_t^j, \\ x_0^j = x_{in}^j, \quad \theta_0^j = \theta_{in}^j, \quad j = 1, \cdots, N. \end{cases} \tag{4.1.3}$$

Moreover, the expectation of sum of heading angles is conserved:

Proposition 4.1.1. *Let (X_t, Θ_t) be the solution of* (1.0.8) *with the initial data $(X_0, \Theta_0) = (X_{in}, \Theta_{in})$. Then, the total sum phase process $\mathcal{S}_t := \sum_{j=1}^N \theta_t^j$ is continuous martingale, i.e., for the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ with respect to (X_t, Θ_t), we have*

$$\mathbb{E}\left[\mathcal{S}_t | \mathcal{F}_s\right] = \mathcal{S}_s, \quad \forall\ t \geq s \geq 0.$$

Therefore, Proposition 4.1.1 suggests that the total sum of the initial positions $\sum_{j=1}^N x_0^j$ and that of the initial heading angles $\mathcal{S}_0$ can be assumed to be zero so that the expectation of the time-evolution of $\mathcal{S}_t$ is also identically zero.

We now introduce several well-known properties of martingale processes to be used in Section 4.3.

Lemma 4.1.1. *Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, and B_t be the standard one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$. Then, the following assertions hold:*

1. *(Doob's martingale inequality): Let $N : [0, \infty) \times \Omega \to [0, \infty)$ be a non-negative continuous submartingale with respect to $(\mathcal{F}_t)_{t\geq 0}$:*

   $$0 \leq N_s \leq \mathbb{E}\left[N_t | F_s\right], \quad \forall\ 0 \leq s \leq t < \infty.$$

   Then, for any positive constant C and time T, we have

   $$\mathbb{P}\left[\left(\sup_{0\leq t\leq T} N_t\right) > C\right] \leq \frac{\mathbb{E}\left[N_T\right]}{C}.$$

2. *(Doob's supermartingale convergence theorem): Let $N : [0, \infty) \times \Omega \to \mathbb{R}$ be a continuous supermartingale with respect to $(\mathcal{F}_t)_{t\geq 0}$:*

   $$N_s \geq \mathbb{E}\left[N_t | F_s\right], \quad \forall\ 0 \leq s \leq t < \infty.$$

   Then, if $\sup_{t>0} \mathbb{E}[N_t^-] < \infty$, the pointwise limit

   $$N(\omega) = \lim_{t\to\infty} N_t(\omega)$$

   exists and is finite for $\mathbb{P}$-almost all $\omega \in \Omega$.

3. *(Law of the iterated logarithm): For almost surely, we have*

$$\limsup_{t\to\infty} \frac{|B_t|}{\sqrt{2t\log\log t}} = 1.$$

4. *(Law of the interated logarithm for martingales): Assume that M_t is a continuous local martingale process. Then, for almost surely, we have*

$$\limsup_{t\to\infty} \frac{|M_t|}{\sqrt{2\langle M_t\rangle \log\log(\langle M_t\rangle)}} = 1,$$

   where $\langle M_t\rangle := \mathbb{E}(M_t^2)$.

These properties are crucially used in our analysis on the behavior of sample paths. In particular, the third and fourth one, the law of the iterated logarithm for Brownian motion and local martingales are used for the analysis of the two-body system in Section 4.2.

On top of that, we recall a stochastic analogue of Barbalat's lemma of sample paths to show certain convergence of stochastic processes from their integrability. We first define absolute integrability and strong uniform continuity in probability as follows:

Definition 4.1.1. [80] *Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and $(p_t)_{t\geq 0}$ be an adapted process.*

1. *The stochastic process $(p_t)_{t\geq 0}$ is said to be absolutely integrable if*

$$\mathbb{E}\left[\int_0^\infty |p_t|dt\right] < \infty.$$

2. *The stochastic process $(p_t)_{t\geq 0}$ is said to be strongly bounded in probability if for any $\varepsilon > 0$ there exists an $r > 0$ such that*

$$\mathbb{P}\left\{\sup_{t\geq 0} |p_t| > r\right\} \leq \varepsilon.$$

3. *The stochastic process $(p_t)_{t\geq 0}$ is said to be strongly uniformly continuous in probability if for any $a, b > 0$ there exists $\delta = \delta(a, b)$ such that the following inequality holds: for any stopping time $\tau \geq 0$,*

$$\mathbb{P}\left\{\{\tau < \infty\} \cap \left\{\sup_{0\leq s\leq\delta} |p_{\tau+s} - p_\tau| > a\right\}\right\} < b.$$

As in the deterministic Barbalat's lemma, we need an alternative notion of the uniform continuity for the stochastic process. In fact, the strong uniform continuity in probability naturally arises for Itô's processes:

Lemma 4.1.2. [80] *Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space.*

1. *(Strongly uniformly continuous in probability): Let x_t be a stochastic solution process of SDE*

$$dx_t = f(x_t, t)dt + g(x_t, t)dB_t,$$

where $x_t \in \mathbb{R}^n$ and $f(x, t), g(x, t)$ are piecewise continuous in t and locally bounded Lipschitz in x uniformly in t. Then, if x_t is also strongly bounded in probability, then $\beta(x_t)$ is strongly uniformly continuous in probability for any continuous function $\beta : \mathbb{R}^n \to \mathbb{R}^m$.

2. *(Stochastic analog of Barbalat's lemma): If a continuous adapted process $(p_t)_{t\geq 0}$ is strongly uniformly continuous in probability and absolutely integrable, then $(p_t)_{t\geq 0}$ converges to zero almost surely.*

Here, the convergence result in Lemma 4.1.2 is used in Section 4.3 in order to guarantee the convergence of $R_2(\Theta) = R_1(2\Theta)$ to 1.

4.2 A two-body system

We prove the convergence of $\theta_1 - \theta_2$ in the two-agent stochastic J-K model ($N = 2$) with multiplicativ noise, which indicates the emergence of the heading angle alignment.

4.2.1 ψ-independent noise

When there are only two agents, the SDE (4.1.3) is represented as follows:

$$
\begin{cases}
dx_t^1 = (\cos\theta_t^1, \sin\theta_t^1)dt, \quad dx_t^2 = (\cos\theta_t^2, \sin\theta_t^2)dt, \\
d\theta_t^1 = \dfrac{\kappa}{2}\psi(\|x_t^2 - x_t^1\|)\sin(\theta_t^2 - \theta_t^1)dt + \dfrac{\sqrt{2\sigma}}{2}\sin(\theta_t^2 - \theta_t^1)dB_t^1, \\
d\theta_t^2 = \dfrac{\kappa}{2}\psi(\|x_t^1 - x_t^2\|)\sin(\theta_t^1 - \theta_t^2)dt + \dfrac{\sqrt{2\sigma}}{2}\sin(\theta_t^1 - \theta_t^2)dB_t^2, \\
x_0^i = x_{in}^i, \quad \theta_0^i = \theta_{in}^i, \quad i = 1, 2.
\end{cases}
$$

For simplicity, we set $x_t := x_t^1 - x_t^2$ and

$$
\theta_t := \frac{\theta_t^1 - \theta_t^2}{2}, \quad \bar{\theta}_t := \frac{\theta_t^1 + \theta_t^2}{2}, \quad B_t := \frac{B_t^1 - B_t^2}{\sqrt{2}}, \quad \overline{B}_t := \frac{B_t^1 + B_t^2}{\sqrt{2}},
$$

so that the processes $(x_t, \theta_t, \tilde{\theta}_t)$ satisfy

$$
\begin{cases}
dx_t = 2\sin\theta_t \cdot (-\sin\bar{\theta}_t, \cos\bar{\theta}_t)\ dt, \quad t > 0, \\
d\theta_t = -\dfrac{\kappa}{2}\psi(\|x_t\|)\sin 2\theta_t dt - \dfrac{\sqrt{\sigma}}{2}\sin 2\theta_t d\overline{B}_t, \\
d\bar{\theta}_t = \dfrac{\sqrt{\sigma}}{2}\sin 2\theta_t dB_t, \\
x_0 = x_{in}^1 - x_{in}^2, \quad \theta_{in} = \dfrac{\theta_{in}^1 - \theta_{in}^2}{2}, \quad \bar{\theta}_{in} = \dfrac{\theta_{in}^1 + \theta_{in}^2}{2}.
\end{cases} \tag{4.2.1}
$$

Since the noise term in $d\theta_t$ vanishes at $\sin 4\theta_t = 0$, one can show that θ_t is uniformly bounded as in [47]. In fact, the following lemma shows that $\sin 4\theta_t$ never touches zero in any finite time if it initially nonzero.

Lemma 4.2.1. *Let $(x_t, \theta_t, \bar{\theta}_t)$ be the solution process of (4.2.1) with the initial data $(x_{in}, \theta_{in}, \bar{\theta}_{in})$ satisfying*

$$
\kappa, \sigma \geq 0, \quad \theta_{in} \in \left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right), \quad n \in \mathbb{Z}.
$$

Then, the relative heading angle process θ_t stays in an open interval $(\frac{n\pi}{2}, \frac{(n+1)\pi}{2})$ almost surely.

Proof. We first define the function Y as

$$Y(x) := \frac{1}{2}\log(|\tan x|).$$

Then, from the Itô's formula, $Y(\theta_t)$ satisfies

$$dY(\theta_t) = \frac{1}{\sin 2\theta_t}d\theta_t - \frac{\cos 2\theta_t}{\sin^2 2\theta_t}(d\theta_t)^2 = -\left(\frac{\kappa}{2}\psi(\|x_t\|) + \frac{\sigma}{4}\cos 2\theta_t\right)dt - \frac{\sqrt{\sigma}}{2}d\overline{B}_t, \tag{4.2.2}$$

for all $t < \tau_0$, where τ_0 is a stopping time defined by the first hitting time to $\frac{n\pi}{2}$ or $\frac{(n+1)\pi}{2}$:

$$\tau_0 := \inf\left\{t > 0 : \theta_t \notin \left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right)\right\}.$$

Now, we claim:

$$\mathbb{P}\{\tau_0 < \infty\} = 0.$$

To prove this, it suffices to show that the probability of τ_0 smaller than any finite number $T > 0$ is zero, i.e.,

$$\mathbb{P}\{\tau_0 < T\} = 0, \quad \forall\, T > 0. \tag{4.2.3}$$

Then, from the Doob's martingale inequality, we have

$$\mathbb{P}\left\{\sup_{0\le t<T} e^{\lambda\overline{B}_t - \frac{\lambda^2}{2}t} > e^h\right\} \le \frac{\mathbb{E}\left[e^{\lambda\overline{B}_T - \frac{\lambda^2}{2}T}\right]}{e^h} = e^{-h} \quad \forall\lambda\in\mathbb{R} \text{ and } h, T > 0,$$

since the process $e^{\lambda\overline{B}_t - \frac{\lambda^2}{2}t}$ is martingale. In other words, for a stopping time $\tau_{\lambda,h}$ defined by

$$\tau_{\lambda,h} := \inf\left\{t > 0 : \lambda\overline{B}_t - \frac{\lambda^2}{2}t > h\right\},$$

we have

$$\mathbb{P}\{\tau_{\lambda,h} < T\} \le e^{-h}, \quad \forall\,\lambda\in\mathbb{R} \text{ and } h, T > 0.$$

Then, by integrating the above formula (4.2.2), we deduce

$$\begin{aligned}
|\lambda Y(\theta_t)| &\le |\lambda Y(\theta_0)| + \frac{|\lambda|}{2}\left|\int_0^t \left(\kappa\psi(\|x_s\|) + \sigma|\cos 2\theta_s|\right)ds\right| + \frac{\sqrt{\sigma}}{2}|\lambda\overline{B}_t| \\
&\le |\lambda Y(\theta_0)| + \frac{|\lambda|}{2}\left(\kappa\psi_{\max} + \sigma\right)t + \frac{\sqrt{\sigma}}{2}|\lambda\overline{B}_t| \\
&\le |\lambda Y(\theta_0)| + \frac{|\lambda|}{2}\left(\kappa\psi_{\max} + \sigma\right)t + \frac{\sqrt{\sigma}}{2}\left(\frac{\lambda^2}{2}t + h\right),
\end{aligned}$$

for all $t < \tau_0 \wedge \tau_{\lambda,h} \wedge \tau_{-\lambda,h}$. Therefore, if we assume $\tau_0 < \tau_{\lambda,h} \wedge \tau_{-\lambda,h} \leq \infty$, there is a finite upper bound of $|Y(\theta_t)|$ at time τ_0 by the continuity of θ_t and Y, which contradicts to the definition of τ_0.

Therefore, for all $\lambda, h > 0$, we have $\tau_0 \geq \tau_{\lambda,h} \wedge \tau_{-\lambda,h}$ and

$$\mathbb{P}\{\tau_0 < T\} \leq \mathbb{P}\{\tau_{\lambda,h} \wedge \tau_{-\lambda,h} < T\} \leq \mathbb{P}\{\tau_{\lambda,h} < T\} + \mathbb{P}\{\tau_{-\lambda,h} < T\} \leq 2e^{-h},$$

and by taking $h \to \infty$, we deduce (4.2.3) to conclude the desired result. □

Below, we characterize the asymptotic behavior of relative heading angle for the two-agent system (4.2.1).

Theorem 4.2.1. *Let* $(x_t, \theta_t, \bar{\theta}_t)$ *be the solution process of* (4.2.1) *with the initial data* $(x_{in}, \theta_{in}, \bar{\theta}_{in})$ *satisfying*

$$\theta_{in} \in \left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right),$$

and suppose that for a positive constant $\psi_{\min}$*, the system parameters satisfy*

$$\psi(x) \geq \psi_{\min} \ \forall\ x \quad \text{and} \quad \kappa\psi_{\min} > \sigma \geq 0. \tag{4.2.4}$$

Then, we have the following convergence results:

1. *If n is even, one has*
$$\lim_{t\to\infty} \theta_t = \frac{n\pi}{2} \quad \text{almost surely.}$$
2. *If n is odd, one has*
$$\lim_{t\to\infty} \theta_t = \frac{(n+1)\pi}{2} \quad \text{almost surely.}$$

Proof. First, as a consequence of Lemma 4.2.1, the stopping time τ_0 is almost surely infinite. Then, we use the Law of iterated logarithm to obtain

$$\begin{aligned}
&\limsup_{t\to\infty} (Y(\theta_t) - Y(\theta_0)) \\
&\leq \limsup_{t\to\infty} \left(-\left(\frac{\kappa}{2}\psi_{\min} - \frac{\sigma}{4}\right)t - \frac{\sqrt{\sigma}}{2}\overline{B}_t\right) \\
&\leq \limsup_{t\to\infty} \sqrt{2t\log\log t}\left(-\frac{\left(\frac{\kappa}{2}\psi_{\min} - \frac{\sigma}{4}\right)t}{\sqrt{2t\log\log t}} - \frac{\sqrt{\sigma}}{2}\frac{\overline{B}_t}{\sqrt{2t\log\log t}}\right) \\
&= -\infty \quad \text{a.s.,}
\end{aligned}$$

where we used the assumption (4.2.4) to determine the sign of linear term $\left(\frac{\kappa}{2}\psi_{\min} - \frac{\sigma}{2}\right) t$. Therefore, we conclude

$$\lim_{t\to\infty} |\tan\theta_t| = 0$$

and the convergence of θ_t to the desired limit almost surely. □

4.2.2 ψ-dependent noise

In the previous case, the system (4.1.3) and (4.2.1) considered the stochastic perturbation of each communication strength $\kappa\psi$. On the other hand, we here consider the universal coupling scale κ as a random process. We formally write the coupling κ into $\kappa + \sqrt{2\sigma}\dot{B}_t$ in (1.0.4) and assume $N = 2$, the noise term now has a dependence on the communication weight ψ as follows:

$$\begin{cases} dx_t = 2\sin\theta_t \cdot (-\sin\bar{\theta}_t, \cos\bar{\theta}_t)\ dt, \quad t > 0, \\ d\theta_t = -\left(\dfrac{1}{2}\psi(\|x_t\|)\sin 2\theta_t\right)\left(\kappa dt + \sqrt{2\sigma}dB_t\right), \quad d\bar{\theta}_t = 0, \\ x_0 = x_{in}^1 - x_{in}^2, \quad \theta_0 = \frac{\theta_{in}^1 - \theta_{in}^2}{2}, \quad \bar{\theta}_0 = \frac{\theta_{in}^1 + \theta_{in}^2}{2}. \end{cases} \tag{4.2.5}$$

Then, we have the following convergence estimate analogous to Theorem 4.2.1.

Theorem 4.2.2. *Let (x_t, θ_t) be a solution process of (4.2.1) with initial data (x_{in}, θ_{in}). If the system parameters κ, σ, ψ and the initial angle θ_{in} satisfy*

$$\theta_{in} \in \left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right), \quad \kappa > \sigma \max_x \psi(x) =: \sigma\psi_M \quad \textit{and} \quad \int_0^\infty \psi(s)^2 ds = \infty,$$

1. If n is even and ψ satisfies

$$\liminf_{t\to\infty} \frac{\int_0^t \psi(s)ds}{\sqrt{F(t)}} > \frac{2\sqrt{\sigma\psi_M^2}}{\kappa - \sigma\psi_M}, \quad F(x) = x\log(\log x),$$

then θ_t converges to $\frac{n\pi}{2}$ almost surely.

2. If n is odd and ψ satisfies

$$\liminf_{t\to\infty} \frac{\int_0^t \psi(s)ds}{\sqrt{F\left(t\right)}} > \frac{\sqrt{2\sqrt{2}\sigma\psi_M^2}}{\kappa - \sigma\psi_M}, \quad F(x) = x\log\left(\log x\right),$$

then θ_t converges to $\frac{(n+1)\pi}{2}$ almost surely.

Proof. We again employ the Itô's formula to obtain $dY(\theta_t)$. Then, we have

$$\begin{aligned} dY(\theta_t) &= \left(\frac{dY}{d\theta}\right) d\theta_t + \frac{1}{2}\left(\frac{d^2Y}{d\theta^2}\right)(d\theta_t)^2 = \frac{1}{\sin 2\theta_t} d\theta_t - \frac{\cos 2\theta_t}{\sin^2 2\theta_t}(d\theta_t)^2 \\ &= -\left(\frac{\kappa}{2}\psi(\|x_t\|) + \frac{\sigma}{2}(\cos 2\theta_t)\psi(\|x_t\|)^2\right) dt - \frac{\sqrt{2\sigma}}{2}\psi(\|x_t\|)dB_t \\ &\le -\left(\frac{\kappa}{2} - \frac{\sigma}{2}\psi_{\max}\right)\psi(\|x_t\|)dt - \frac{\sqrt{2\sigma}}{2}\psi(\|x_t\|)dB_t \quad \forall\ t < \tau_0, \end{aligned}$$

where τ_0 is a stopping time defined as in Lemma 4.2.1.

In order to estimate the noise term, we define a new Itô process M_t characterized by

$$dM_t = -\frac{\sqrt{2\sigma}}{2}\psi(\|x_t\|)dB_t, \quad M_0 = 0.$$

Then, by the similar argument to Lemma 4.2.1, we have $\tau_0 = \infty$ almost surely, using the exponential martingale of M_t instead of B_t.

Therefore, the stochastic process Y now satisfies

$$\limsup_{t\to\infty} Y(\theta_t) \le Y(\theta_0) + \limsup_{t\to\infty}\left(-\left(\frac{\kappa}{2} - \frac{\sigma}{2}\psi_{max}\right)\int_0^t \psi(\|x_s\|)ds + |M_t|\right),$$

and from Lemma 4.1.1, we also have an estimate of M_t:

$$\limsup_{t\to\infty} \frac{|M_t|}{\sqrt{2F\left(\langle M_t\rangle\right)}} = \limsup_{t\to\infty} \frac{|M_t|}{\sqrt{2F\left(\int_0^t \frac{\sigma\psi_s^2}{2}ds\right)}} = 1,$$

where we used a notation $\psi_s := \psi(\|x_s\|)$ for simplicity.

Hence, whenever

$$\limsup_{t\to\infty} \frac{-\left(\frac{\kappa}{2}-\frac{\sigma}{2}\psi_{max}\right)\int_0^t \psi_s ds + |M_t|}{\sqrt{2F\left(\int_0^t \frac{\sigma\psi_s^2}{2}ds\right)}} = -\liminf_{t\to\infty} \frac{\left(\frac{\kappa}{2}-\frac{\sigma}{2}\psi_{max}\right)\int_0^t \psi_s ds}{\sqrt{2F\left(\int_0^t \frac{\sigma\psi_s^2}{2}ds\right)}} + 1 \tag{4.2.6}$$

is negative and

$$\int_0^\infty \psi_s^2 ds = \int_0^\infty \psi(\|x_s\|)^2 ds = \infty,$$

we have

$$\limsup_{t\to\infty} Y(\theta_t) = -\infty \quad \text{and} \quad \lim_{t\to\infty} |\tan\theta_t| = 0,$$

and conclude the almost surely convergence of θ_t.

Let us verify the condition for (4.2.6). If n is even, we can estimate the upper bound of $\|x_t\|$. More precisely, we have

$$\|x_t\| \le \|x_0\| + \left\|\int_0^t (0, 2\sin\theta_s)ds\right\| \le \|x_0\| + \sqrt{2}t,$$

and

$$\int_0^t \psi_s ds \ge \int_0^t \psi(\|x_0\| + \sqrt{2}s)ds = \frac{1}{\sqrt{2}}\int_{\|x_0\|}^{\|x_0\|+\sqrt{2}t} \psi(s)ds, \quad \int_0^t \psi_s^2 ds \le t\psi_M^2.$$

Similarly, if n is odd, we have

$$(-\|x_0\| + \sqrt{2}t)_+ \le \|x_t\| \le \|x_0\| + \left\|\int_0^t (0, 2\sin\theta_s)ds\right\| \le \|x_0\| + 2t,$$

and

$$\int_0^t \psi_s ds \ge \frac{1}{2}\int_{\|x_0\|}^{\|x_0\|+2t} \psi(s)ds, \quad \int_0^t \psi_s^2 ds \le \frac{\|x_0\|}{\sqrt{2}}\psi_M^2 + \frac{1}{\sqrt{2}}\int_0^{|\sqrt{2}t-\|x_0\||} \psi(s)^2 ds.$$

Therefore, we substitute these bounds to obtain

$$\liminf_{t\to\infty} \frac{\left(\frac{\kappa}{2}-\frac{\sigma}{2}\psi_{max}\right)\int_0^t \psi_s ds}{\sqrt{2F\left(\int_0^t \frac{\sigma\psi_s^2}{2}ds\right)}} \ge \liminf_{t\to\infty} \frac{\left(\frac{\kappa}{2}-\frac{\sigma}{2}\psi_{max}\right)\frac{1}{\sqrt{2}}\int_0^{\sqrt{2}t} \psi(s)ds}{\sqrt{\sigma\psi_M^2 F(t)}} > 1,$$

for even n, and

$$\liminf_{t\to\infty} \frac{\left(\frac{\kappa}{2} - \frac{\sigma}{2}\psi_{max}\right)\int_0^t \psi_s ds}{\sqrt{2F\left(\int_0^t \frac{\sigma\psi_s^2}{2} ds\right)}} \geq \liminf_{t\to\infty} \frac{\left(\frac{\kappa}{2} - \frac{\sigma}{2}\psi_{max}\right)\frac{1}{2}\int_0^{2t} \psi(s) ds}{\sqrt{\frac{\sigma}{\sqrt{2}}F\left(\int_0^{\sqrt{2}t} \psi(s)^2 ds\right)}} > 1,$$

for odd n. □

4.3 A many-body system

We consider the stochastic J-K model (1.0.8)–(1.0.9) under the all-to-all constant communication weight, $\psi \equiv 1$.

4.3.1 Many-body system with independent white noises

Since the dynamics of θ_t is now decoupled from that of x_t, we may ignore the position process x_t at this moment and only focus on the dynamics of phase process θ_t:

$$\begin{cases} d\theta_t^j = \left(\frac{1}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^j)\right)\left(\kappa dt + \sqrt{2\sigma} dB_t^j\right), \\ \theta_0^j = \theta_{in}^j, \quad j = 1, \cdots, N. \end{cases} \tag{4.3.1}$$

Now, we provide a basic estimate for the order parameter $R(\Theta_t)$ for later use.

Lemma 4.3.1. *Let $\Theta_t = (\theta_t^1, \cdots, \theta_t^N)$ be the solution process of SDE (4.3.1). Then, the order parameters $R_t := R(\Theta_t)$ and $\phi_t := \phi(\Theta_t)$ satisfy*

$$dR_t^2 = \left[2\kappa R_t^2 P_t + 2\sigma R_t^2 \frac{P_t}{N} - 2\sigma R_t^3 \frac{1}{N}\sum_{j=1}^N \sin^2(\phi_t - \theta_t^j)\cos(\phi_t - \theta_t^j)\right] dt$$
$$+ 2\sqrt{2\sigma} R_t^2 \frac{1}{N}\sum_{j=1}^N \sin^2(\phi_t - \theta_t^j) dB_t^j,$$

where the process P_t is given by

$$P_t := \frac{1}{N}\sum_{i=1}^N \sin^2(\theta_t^i - \phi_t).$$

Proof. From the definition of order parameters, (4.3.1) can also be written as

$$d\theta_t^j = R_t \sin(\phi_t - \theta_t^j)(\kappa dt + \sqrt{2\sigma} dB_t^j). \tag{4.3.2}$$

Then, recall that for a solution process to the stochastic differential equation

$$d\mathbf{X}_t = \mu_t dt + \mathbf{G}_t d\mathbf{B}_t$$

and C^2 function $\mathbf{F}(\cdot)$, the Itô's formula reads as below:

$$\begin{aligned}
d\mathbf{F}(\mathbf{X}_t) &= \left(\frac{d\mathbf{F}}{d\mathbf{X}}\right)^T d\mathbf{X}_t + \frac{1}{2}(d\mathbf{X}_t)^T \left(\frac{d^2\mathbf{F}}{d\mathbf{X}^2}\right)(d\mathbf{X}_t) \\
&= \left[\left(\frac{d\mathbf{F}}{d\mathbf{X}}\right)^T \mu_t + \frac{1}{2}\mathrm{Tr}\left[\mathbf{G}_t^T \left(\frac{d^2\mathbf{F}}{d\mathbf{X}^2}\right)\mathbf{G}_t\right]\right] dt + \left(\frac{d\mathbf{F}}{d\mathbf{X}}\right)^T \mathbf{G}_t d\mathbf{B}_t \\
&= \left[\left(\frac{d\mathbf{F}}{d\mathbf{X}}\right)^T \mu_t + \frac{1}{2}\mathrm{Tr}\left[\mathbf{G}_t \mathbf{G}_t^T \left(\frac{d^2\mathbf{F}}{d\mathbf{X}^2}\right)\right]\right] dt + \left(\frac{d\mathbf{F}}{d\mathbf{X}}\right)^T \mathbf{G}_t d\mathbf{B}_t.
\end{aligned}$$

Now, recall that the order parameter satisfy

$$R^2(\Theta_t) = \frac{1}{N^2}\sum_{j,k=1}^N \cos(\theta_t^k - \theta_t^j).$$

Then, the first and second derivatives of R^2 are

$$\begin{aligned}
\frac{dR^2}{d\theta^j} &= \frac{2}{N^2}\sum_{k=1}^N \sin(\theta^k - \theta^j) = \frac{2R}{N}\sin(\phi - \theta^j), \\
\frac{d^2R^2}{(d\theta^j)^2} &= \frac{2}{N^2}\sum_{k\neq j}(-\cos(\theta^k - \theta^j)) = \frac{2}{N^2}\left(-NR\cos(\phi - \theta^j) + 1\right).
\end{aligned}$$

Since $\mathbf{G}_t$ is now a diagonal matrix, Itô's formula yields

$$\begin{aligned}
dR_t^2 &= \left[\sum_{j=1}^{N}\left(\frac{dR^2}{d\theta^j}\right)\kappa R_t\sin(\phi_t-\theta_t^j)+\sigma\sum_{j=1}^{N}\big(R_t\sin(\phi_t-\theta_t^j)\big)^2\left(\frac{d^2R^2}{(d\theta^j)^2}\right)\right]dt\\
&\quad+\sum_{j=1}^{N}\left(\frac{dR^2}{d\theta^j}\right)\sqrt{2\sigma}R_t\sin(\phi_t-\theta_t^j)dB_t^j\\
&=\left[\kappa R_t\sum_{j=1}^{N}\sin(\phi_t-\theta_t^j)\left(\frac{dR^2}{d\theta^j}\right)+\sigma R_t^2\sum_{j=1}^{N}\sin^2(\phi_t-\theta_t^j)\left(\frac{d^2R^2}{(d\theta^j)^2}\right)\right]dt\\
&\quad+\sum_{j=1}^{N}\frac{R_t}{N}\sin(\phi_t-\theta_t^j)\cdot\sqrt{2\sigma}R_t\sin(\phi_t-\theta_t^j)dB_t^j\\
&=\left[2\kappa R_t^2\frac{1}{N}\sum_{j=1}^{N}\sin^2(\phi_t-\theta_t^j)\right]dt\\
&\quad+\left[2\sigma R_t^2\frac{1}{N^2}\sum_{j=1}^{N}\sin^2(\phi_t-\theta_t^j)\big(-NR_t\cos(\phi_t-\theta_t^j)+1\big)\right]dt\\
&\quad+2\sqrt{2\sigma}R_t^2\frac{1}{N}\sum_{j=1}^{N}\sin^2(\phi_t-\theta_t^j)dB_t^j\\
&=\left[2\kappa R_t^2P_t+2\sigma R_t^2\frac{P_t}{N}-2\sigma R_t^3\frac{1}{N}\sum_{j=1}^{N}\sin^2(\phi_t-\theta_t^j)\cos(\phi_t-\theta_t^j)\right]dt\\
&\quad+2\sqrt{2\sigma}R_t^2\frac{1}{N}\sum_{j=1}^{N}\sin^2(\phi_t-\theta_t^j)dB_t^j.
\end{aligned}$$

□

Note that the drift and diffusion terms in the above Lemma are continuous functions proportional to R_t^2. Therefore, we may consider a formula for the logarithm of R_t^2 until $R_t = 0$ for some $t > 0$. In the following lemma, we show that whenever $R(\Theta_{in})$ is strictly positive, R_t stays positive in finite time almost surely.

Lemma 4.3.2. *Let $\Theta_t = (\theta_t^1, \cdots, \theta_t^N)$ be a solution process of* (4.3.1)*. Then, if the initial order parameter $R_{in} = R(\Theta_{in})$ is positive, the following assertions hold.*

1. *For every $\kappa, \sigma \geq 0$, we have*

$$\tau_R := \inf\{t > 0 : R_t = 0\} = \infty \ \textit{a.s.}$$

2. *If we further assume $\kappa > \sigma\left(1 + \frac{1}{N}\right) > 0$, we have*

$$\int_0^\infty \mathbb{E}\left[\sin^2(\theta_t^i - \phi_t)\right] dt < \infty, \quad i = 1, \cdots, N.$$

Proof. (1) From Lemma 4.3.1, we have

$$\frac{(dR_t^2)(dR_t^2)}{2R_t^4} = 4\sigma\left(\frac{1}{N}\sum_{j=1}^N \sin^2(\phi_t - \theta_t^j)dB_t^j\right)^2 = \frac{4\sigma}{N^2}\sum_{j=1}^N \sin^4(\phi_t - \theta_t^j)dt,$$

so that the log-derivative of R^2 satisfies

$$\begin{aligned} d\log(R_t^2) &= \frac{dR_t^2}{R_t^2} - \frac{(dR_t^2)(dR_t^2)}{2R^4} \\ &= \left[2\kappa P_t + \frac{2\sigma P_t}{N} - \frac{2\sigma R_t}{N}\sum_{j=1}^N \sin^2(\phi_t - \theta_t^j)\cos(\phi_t - \theta_t^j)\right] dt \\ &\quad - \left[\frac{4\sigma}{N^2}\sum_{j=1}^N \sin^4(\phi_t - \theta_t^j)\right] dt \\ &\quad + \frac{2\sqrt{2\sigma}}{N}\sum_{j=1}^N \sin^2(\phi_t - \theta_t^j)dB_t^j, \quad \forall\ t < \tau_R. \end{aligned} \tag{4.3.3}$$

Now, consider a continuous martingale process $(m_t)_{t\geq 0}$ that estimates the noise term in $d\log(R_t^2)$ as

$$dm_t = \frac{2\sqrt{2\sigma}}{N}\sum_{j=1}^N \sin^2(\phi_t - \theta_t^j)dB_t^j, \quad m_0 = 0,$$

and we use Doob's martingale inequality as in Lemma 4.2.1 to obtain

$$\mathbb{P}\left\{\tau^m_{\lambda,h} < T\right\} \le e^{-h}, \quad \forall \lambda \in \mathbb{R} \text{ and } h, T > 0,$$

where the stopping time $\tau^m_{\lambda,h}$ is defined by

$$\tau^m_{\lambda,h} := \inf\left\{t > 0 : \lambda m_t - \frac{\lambda^2}{2}\langle m_t\rangle > h\right\}.$$

Then, by integrating (4.3.3), one can deduce

$$\begin{aligned}
&\log(R_t^2) - \log(R_0^2) \\
&= \int_0^t \left(2\kappa P_s + \frac{2\sigma P_s}{N}\right) ds \\
&- \int_0^t \left(\frac{2\sigma R_s}{N}\sum_{j=1}^N \sin^2(\phi_s - \theta_s^j)\cos(\phi_s - \theta_s^j) + \frac{4\sigma}{N^2}\sum_{j=1}^N \sin^4(\phi_s - \theta_s^j)\right) ds + m_t \\
&\ge -\left(2\sigma + \frac{4\sigma}{N}\right) t + \frac{\lambda}{2}\langle m_t\rangle + \frac{h}{\lambda} \\
&= -\left(2\sigma + \frac{4\sigma}{N}\right) t + \frac{4\lambda\sigma}{N^2}\int_0^t \sum_{j=1}^N \sin^4(\phi_s - \theta_s^j) ds + \frac{h}{\lambda} \\
&\ge -\left(2\sigma + \frac{4\sigma}{N}\right) t + \frac{h}{\lambda}, \quad \forall\, t < \tau_R \wedge \tau^m_{\lambda,h}, \quad \lambda < 0,\ h > 0.
\end{aligned}$$

Therefore, if $\tau_R < \tau^m_{\lambda,h} \le \infty$, there is a finite lower bound of $\log(R_t^2)$ at time τ_R from the continuity of R_t. This is a contradiction, and we have

$$\mathbb{P}\{\tau_R < T\} \le \mathbb{P}\left\{\tau^m_{\lambda,h} < T\right\} \le e^{-h}, \quad \forall\, \lambda < 0,\ h, T > 0,$$

and deduce $\mathbb{P}\{\tau_R < T\} = 0$ for all finite T by taking $h \to \infty$ and conclude the desired result.

(2) We integrate $d\log(R_t^2)$ to get

$$\begin{aligned}
\mathbb{E}\left[\log(R_t^2) - \log(R_0^2)\right] &\ge \mathbb{E}\left[\int_0^t \left(2\kappa P_s + \frac{2\sigma P_s}{N} - 2\sigma P_s - \frac{4\sigma}{N}P_s\right) ds\right] \\
&\ge \left(2\kappa - 2\sigma\left(1 + \frac{1}{N}\right)\right)\mathbb{E}\left[\int_0^t P_s ds\right].
\end{aligned}$$

Therefore, since $-\log(R_t^2)$ is always nonnegative, we have

$$\int_0^t \mathbb{E}[P_s]ds = \mathbb{E}\left[\int_0^t P_s ds\right] \leq \frac{-\log(R_0^2)}{2\kappa - 2\sigma\left(1+\frac{1}{N}\right)}, \quad \forall t \geq 0,$$

which yields the desired result. □

Now, we are ready to present our second main result on the asymptotic convergence of the processes R_t and P_t.

Theorem 4.3.1. *Let $\theta_t = (\theta_t^1, \cdots, \theta_t^N)$ be a solution process of (4.3.1). If the system parameters κ, σ and initial phase $\Theta_{in} = (\theta_{in}^1, \cdots, \theta_{in}^N)$ satisfy*

$$\kappa > \sigma\left(1 + \frac{1}{N}\right) \quad \textit{and} \quad R(\Theta_{in}) > 0,$$

there exist a random variable $r_\infty > 0$ such that

$$\lim_{t\to\infty} R_2(\Theta_t) = 1, \quad \lim_{t\to\infty} P_t = 0 \quad \textit{and} \quad \lim_{t\to\infty} R(\Theta_t) = r_\infty > 0, \quad \textit{a.s.}$$

Proof. (i) (Convergence of $R_2(\Theta_t)$): In order to apply Lemma 4.1.2 (3), P_t has to be absolutely integrable and the strong uniformly continuous in probability. On one hand, Lemma 4.3.2 guarantees the absolute integrability of P_t in Definition 4.1.1:

$$\mathbb{E}\int_0^\infty |P_t| ds < \infty.$$

On the other hand, there is a technical difficulty to show the strong uniform continuity in probability of P_t, since $\phi_t = \phi(\Theta_t)$ and P_t are not continuous in terms of θ_t. However, P_t has a lower bound that can be described without ϕ_t but using the order parameter R_2 of Θ_t:

$$\begin{aligned} P_t &= \frac{1}{N}\sum_{j=1}^N \sin^2(\theta_t^j - \phi_t) = \frac{1}{2} - \frac{1}{2N}\sum_{j=1}^N \cos(2\theta_t^j - 2\phi_t) \\ &= \frac{1}{2} - \frac{1}{2}R_2(\Theta_t)\cos(2\phi_2(\Theta_t) - 2\phi(\Theta_t)) \geq \frac{1 - R_2(\Theta_t)}{2}, \end{aligned}$$

which implies that $1 - R_2(\Theta_t)$ is absolutely integrable. Moreover, from (4.1.1), we also have

$$R_2(\Theta_t)e^{\mathrm{i}(2\phi_2(\Theta_t) - 2\theta_t^k)} = \frac{1}{N}\sum_{j=1}^N e^{\mathrm{i}(2\theta_t^j - 2\theta_t^k)},$$

and by averaging both sides over k, we conclude

$$(R_2(\Theta_t))^2 = \frac{1}{N^2}\sum_{j,k=1}^{N} e^{\mathrm{i}(2\theta_t^j - 2\theta_t^k)} = \frac{1}{N^2}\sum_{j,k=1}^{N}\cos(2\theta_t^j - 2\theta_t^k).$$

Note that $(\cos\theta_t^j, \sin\theta_t^j)$ is strongly uniformly continuous in probability from Lemma 4.1.2 (1), since it is bounded and also a solution process of SDEs

$$\begin{aligned}
d\cos\theta_t^j &= -\left(R_t \sin\theta_t^j \sin(\phi_t - \theta_t^j)\right)\left(\tilde{\kappa}dt + \sqrt{2\tilde{\sigma}}dB_t^j\right) \\
&\quad - \tilde{\sigma}R_t^2 \cos\theta_t^j \sin^2(\phi_t - \theta_t^j)dt, \\
d\sin\theta_t^j &= \left(R_t \cos\theta_t^j \sin(\phi_t - \theta_t^j)\right)\left(\tilde{\kappa}dt + \sqrt{2\tilde{\sigma}}dB_t^j\right) \\
&\quad - \tilde{\sigma}R_t^2 \sin\theta_t^j \sin^2(\phi_t - \theta_t^j)dt, \\
\theta_0^j &= \theta_{in}^j, \quad j = 1, \cdots, N,
\end{aligned} \tag{4.3.4}$$

where $R_t \sin(\phi_t - \theta_t^j) = \frac{1}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^j)$ is an analytic function of $2N$-vector

$$(\cos\theta_t^1, \sin\theta_t^1, \cdots, \cos\theta_t^N, \sin\theta_t^N).$$

Hence, $R_2(\Theta_t)$ is a continuous function of $\{\cos\theta_t^j\}_{j=1}^N$ and $\{\sin\theta_t^j\}_{j=1}^N$ and it is strongly uniformly continuous in probability from Lemma 4.1.2 (1). Therefore, $1 - R_2(\Theta_t)$ converges to zero almost surely.

(ii) For the convergence of order parameter R_t, we apply Doob's supermartingale convergence theorem (see Lemma 4.1.1) for $N_t := -\log(R_t^2)$. Then, since N_t is nonnegative supermartingale, the pathwise limit

$$S(\omega) = \lim_{t\to\infty} -\log(R_t^2(\omega))$$

exists and finite for almost all ω, i.e., $r_\infty(\omega) := e^{-\frac{S(\omega)}{2}} = \lim_{t\to\infty} R_t(\omega)$ exists and finite almost surely.

(iii) (Convergence of P_t): First, recall that $R_2(\Theta_t)$ satisfies

$$R_2^2(\Theta_t) = \frac{1}{N^2}\sum_{i,j=1}^{N}\cos(2\theta_t^i - 2\theta_t^j) = 1 - 2\frac{1}{N^2}\sum_{i,j=1}^{N}\sin^2(\theta_t^i - \theta_t^j).$$

Then, since $R_2(\Theta_t)$ converges to 1 and R_t converges to a positive random variable, we have

$$\sin(\theta_t^j - \phi_t) = \frac{1}{NR_t}\sum_{k=1}^{N}\sin(\theta_t^j - \theta_t^k) \to 0, \quad j = 1, \cdots, N.$$

Therefore,

$$P_t = \frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_t^j - \phi_t)$$

converges to zero almost surely. □

4.3.2 Many-body system with identical white noise

We now consider another multiplicative noise J-K model with identical white noises, which is motivated from the same philosophy to ψ-dependent noise for two-particle system.

Consider the stochastic J-K model with the random universal coupling scale

$$\kappa dt = \bar{\kappa} dt + \sqrt{2\sigma} dB_t$$

as in Section 3.2, and assume $\psi \equiv 1$. Then, we have the following system of SDEs, which had been studied in [47] as a stochastic Kuramoto model with multiplicative noises:

$$\begin{cases} d\theta_t^j = \left(\frac{1}{N}\sum_{k=1}^{N}\sin(\theta_t^k - \theta_t^j)\right)\left(\bar{\kappa} dt + \sqrt{2\sigma} dB_t\right), \\ \theta_0^j = \theta_{in}^j, \quad j = 1, \cdots, N. \end{cases} \tag{4.3.5}$$

One of the main property of (4.3.5) that differs from (4.3.1) is the order preserving property of $\Theta_t = (\theta_t^1, \cdots, \theta_t^N)$. Since noise terms in $d\theta^i$ and $d\theta^j$ are same whenever $\theta^i = \theta^j$ modulo 2π, we have the following result.

Lemma 4.3.3. *[47] Let $\theta_t := (\theta_t^1, \cdots, \theta_t^N)$ be a solution process of* (4.3.5) *with the initial data θ_{in} satisfying*

$$\theta_{in}^i < \theta_{in}^j,$$

for some i, j. Then, we have

$$\theta_t^i \le \theta_t^j, \quad t \ge 0.$$

Therefore, from Lemma 4.3.3 and the construction of θ_t, we may assume

$$\theta_t^1 \le \cdots \le \theta_t^N \le \theta_t^1 + 2\pi, \quad \forall t \ge 0,$$

without loss of generality.

For this alternative model, we first check if there is an analogous result of Theorem 4.3.1.

Theorem 4.3.2. *Let $\Theta_t = (\theta_t^1, \cdots, \theta_t^N)$ be a solution process of (4.3.5). If the system parameters κ, σ and the initial phase $\Theta_{in} = (\theta_{in}^1, \cdots, \theta_{in}^N)$ satisfy*

$$\kappa > \sigma \quad \textit{and} \quad R(\Theta_{in}) > 0,$$

there exists a positive random variable r_∞ such that

$$\lim_{t\to\infty} R_2(\Theta_t) = 1, \quad \lim_{t\to\infty} P_t = 0 \quad \textit{and} \quad \lim_{t\to\infty} R(\Theta_t) = r_\infty > 0, \quad \textit{a.s.}$$

Proof. First, by using Itô's formula to R_t^2, one can derive a log derivative of R_t^2 as in Lemma 4.3.2:

$$\begin{aligned} d\log(R_t^2) = &\left[2\kappa P_t + 2\sigma\left\{-P_t^2 + Q_t^2 - \frac{R_t}{N}\sum_{i=1}^N \sin^2(\theta_t^i - \phi_t)\cos(\theta_t^i - \phi_t)\right\}\right] dt \\ &+ 2\sqrt{2\sigma} P_t dB_t, \end{aligned}$$

where the processes P_t, Q_t and R_t are given by

$$P_t := \frac{1}{N}\sum_{i=1}^N \sin^2(\theta_t^i - \phi_t), \quad Q_t := \frac{1}{N}\sum_{i=1}^N \cos(\theta_t^i - \phi_t)\sin(\theta_t^i - \phi_t), \quad R_t = R(\Theta_t).$$

Then, by using similar stopping time argument to Lemma 4.3.2, one can show that the order parameter $R_t = R(\Theta_t)$ is always positive almost surely. Now, we claim:

$$P_t - P_t^2 - \frac{R_t}{N}\sum_{i=1}^N \sin^2(\theta_t^i - \phi_t)\cos(\theta_t^i - \phi_t) \ge 0. \tag{4.3.6}$$

To prove (4.3.6), we first recall that the order parameter R_t can be written as

$$R_t = \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}(\theta_t^j - \phi_t)} = \mathcal{R}e\left[\frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}(\theta_t^j - \phi_t)}\right] = \frac{1}{N}\sum_{j=1}^{N}\cos(\theta_t^j - \phi_t). \quad (4.3.7)$$

Then, we can show the above inequality (4.3.6) by direct calculation:

$$\begin{aligned}
&\frac{R_t}{N}\sum_{i=1}^{N}\sin^2(\theta_t^i - \phi_t)\cos(\theta_t^i - \phi_t) \\
&\quad = \frac{1}{N^2}\sum_{i,j}\sin^2(\theta_t^i - \phi_t)\cos(\theta_t^i - \phi_t)\cos(\theta_t^j - \phi_t) \\
&\quad \le \frac{1}{2N^2}\sum_{i,j}\sin^2(\theta_t^i - \phi_t)\left(\cos^2(\theta_t^i - \phi_t) + \cos^2(\theta_t^j - \phi_t)\right) \\
&\quad \le \frac{1}{2N^2}\sum_{i,j}\sin^2(\theta_t^i - \phi_t)\left(\cos^2(\theta_t^j - \phi_t) + \cos^2(\theta_t^j - \phi_t)\right) \\
&\quad = \left(\frac{1}{N}\sum_{i=1}^{N}\sin^2(\theta_t^i - \phi_t)\right)\left(\frac{1}{N}\sum_{i=1}^{N}\cos^2(\theta_t^j - \phi_t)\right) \\
&\quad = P_t(1 - P_t),
\end{aligned}$$

where the first inequality uses the Young's inequality $a^2 + b^2 \ge 2ab$, and the second one comes from the Chebyshev's sum inequality:

$$a_1 \le \cdots \le a_N, \quad b_1 \ge \cdots \ge b_N \;\Rightarrow\; \frac{1}{N}\sum_{k=1}^{N} a_k b_k \le \left(\frac{1}{N}\sum_{k=1}^{N} a_k\right)\left(\frac{1}{N}\sum_{k=1}^{N} b_k\right).$$

Then, we put (4.3.6) into the log-derivative formula above to obtain

$$d\log(R_t^2) \ge (2\kappa - 2\sigma)\, P_t dt + 2\sqrt{2\sigma} P_t dB_t,$$

and since $\log(R_t^2)$ is always nonpositive, we have

$$\int_0^t \mathbb{E}[P_s]ds = \mathbb{E}\left[\int_0^t P_s ds\right] \le \frac{-\log(R_0^2)}{2\kappa - 2\sigma}, \quad \forall t \ge 0,$$

which yields the absolute integrability of P_t as in Theorem 4.3.1. Therefore, we apply the same argument with Theorem 4.3.1 (ii), (iii) to conclude the desired results. □

Finally, we present another main result on the convergence of θ_t. Namely, if initial heading angles $\{\theta^i_{in}\}_{i=1}^N$ are confined in a small arc, then one can show that the heading angle alignment state $\theta^1 = \theta^2 = \cdots = \theta^N$ is stable with a high probability.

Theorem 4.3.3. *Let $\Theta_t = (\theta^1_t, \cdots, \theta^N_t)$ be a solution process of* (4.3.1)*. Suppose that the system parameters κ, σ and the initial data Θ_{in} satisfy*

$$\kappa, \sigma > 0, \quad 0 < L_0 < L_\infty < \frac{\pi}{2}, \quad D(\Theta_{in}) < 2L_0.$$

Then, one has a stochastic stability of heading angle alignment with high probability:

$$\mathbb{P}\left\{\sup_{0\le t<\infty} D(\Theta_t) < 2L_\infty, \quad \limsup_{t\to\infty} D(\Theta_t) = 0\right\} \ge 1 - \left(\frac{\sin^2 L_0}{\sin^2 L_\infty}\right)^{\frac{\kappa+\sigma}{2\sigma}\cos^2 L_\infty}.$$

Proof. Without loss of generality, assume that θ^N_{in} and θ^1_{in} are the maximal and minimal angles among all initial phases. From the order preserving property, the maximal difference between angles $\{\theta^j_t\}_{j=1}^N$ is $\theta^N_t - \theta^1_t$ for all $t \ge 0$. In this way, the initial condition $D(\Theta_{in}) < 2L_0$ is equivalent to $\theta^N_{in} - \theta^1_{in} < 2L_0 < \pi$.

Then, we set a process V_t as

$$V_t := 1 - \cos\left(\theta^N_t - \theta^1_t\right) = 1 - \cos D(\Theta_t),$$

and apply Itô's lemma to see

$$\begin{aligned}
d(\log V_t) &= \frac{\sin(\theta^N_t - \theta^1_t)}{1-\cos(\theta^N_t - \theta^1_t)}(d(\theta^N - \theta^1_t)) - \frac{1}{2(1-\cos(\theta^N_t - \theta^1_t))}(d(\theta^N_t - \theta^1_t))^2 \\
&= \frac{\cos\left(\frac{\theta^N_t - \theta^1_t}{2}\right)}{\sin\left(\frac{\theta^N_t - \theta^1_t}{2}\right)}(d(\theta^N_t - \theta^1_t)) - \frac{1}{4\sin^2\left(\frac{\theta^N_t - \theta^1_t}{2}\right)}(d(\theta^N_t - \theta^1_t))^2 \\
&= -2R_t \cos\left(\frac{\theta^N_t + \theta^1_t}{2} - \phi_t\right)\cos\left(\frac{\theta^N_t - \theta^1_t}{2}\right)(\kappa dt + \sqrt{2\sigma} dB_t) \\
&\quad - R_t^2 \cos^2\left(\frac{\theta^N_t + \theta^1_t}{2} - \phi_t\right)(2\sigma dt)
\end{aligned}$$

$$
\begin{aligned}
&= \left[-2\kappa R_t \cos\left(\frac{\theta_t^N+\theta_t^1}{2}-\phi_t\right)\cos\left(\frac{\theta_t^N-\theta_t^1}{2}\right)\right]dt \\
&\quad - \left[2\sigma R_t^2 \cos^2\left(\frac{\theta_t^N+\theta_t^1}{2}-\phi_t\right)\right]dt \\
&\quad - 2\sqrt{2\sigma}R_t \cos\left(\frac{\theta_t^N+\theta_t^1}{2}-\phi_t\right)\cos\left(\frac{\theta_t^N-\theta_t^1}{2}\right)dB_t.
\end{aligned}
$$

Now, we introduce a stopping time τ as the first hitting time of $\theta_t^N-\theta_t^1$ to $2L_\infty$:

$$
\tau := \inf\left\{t>0 : \theta_t^N-\theta_t^1 > 2L_\infty\right\} \quad \text{and} \quad R_\infty := \cos L_\infty.
$$

Then, for $t<\tau$, one has

$$
\cos\left(\frac{\theta_t^N-\theta_t^1}{2}\right) \ge R_\infty, \quad R_t\cos\left(\frac{\theta_t^N+\theta_t^1}{2}-\phi_t\right) \ge R_\infty,
$$

where the second inequality comes from

$$
R_t\cos\left(\frac{\theta_t^N+\theta_t^1}{2}-\phi_t\right) = \frac{1}{N}\sum_{i=1}^N \cos\left(\frac{\theta_t^1+\theta_t^N}{2}-\theta_t^i\right) \ge \cos\left(\frac{\theta_t^N-\theta_t^1}{2}\right).
$$

Next, we estimate the noise term of $d(\log V_t)$. From the equation of $\log V_t$, we define a process M_t by

$$
dM_t = -2\sqrt{2\sigma}R_t\cos\left(\frac{\theta_t^N+\theta_t^1}{2}-\phi_t\right)\cos\left(\frac{\theta_t^N-\theta_t^1}{2}\right)dB_t, \quad t>0, \quad M_0=0.
$$

Then, for any positive real number λ, the process $\exp(\lambda M_t - \frac{\lambda^2}{2}\int_0^t D_s^2 ds)$ is nonnegative continuous martingale process, where

$$
D_t := -2\sqrt{2\sigma}R_t\cos\left(\frac{\theta_t^N+\theta_t^1}{2}-\phi_t\right)\cos\left(\frac{\theta_t^N-\theta_t^1}{2}\right).
$$

Therefore, by using Lemma 4.1.1 (1), one has

$$
\mathbb{P}\left[\left(\sup_{0\le t\le T} e^{\lambda M_t - \frac{\lambda^2}{2}\int_0^t D_s^2 ds}\right) > e^h\right] \le \mathbb{E}\left[e^{\lambda M_T - \frac{\lambda^2}{2}\int_0^T D_s^2 ds}\right]e^{-h} = e^{-h}, \ \forall\, T<\infty.
$$

For fixed $\lambda>0$ and $h>0$, let $\tau_{\lambda,h}$ be a stopping time corresponding to the above inequality,

$$
\tau_{\lambda,h} := \inf\left\{t>0 : \lambda M_t - \frac{\lambda^2}{2}\int_0^t D_s^2 ds > h\right\}.
$$

By definition of $\tau_{\lambda,h}$, one has

$$\mathbb{P}\left\{\tau_{\lambda,h}=\infty\right\}\geq 1-e^{-h},$$

and for $t<\tau\wedge\tau_{\lambda,h}$, we have

$$\begin{aligned}
&\log V_t-\log V_0\\
&=M_t-\int_0^t\left[2\kappa R_s\cos\left(\frac{\theta_s^N+\theta_s^1}{2}-\phi_s\right)\cos\left(\frac{\theta_s^N-\theta_s^1}{2}\right)\right]ds\\
&\qquad-\int_0^t\left[2\sigma R_s^2\cos^2\left(\frac{\theta_s^N+\theta_s^1}{2}-\phi_s\right)\right]ds\\
&\leq M_t-(2\kappa R_\infty^2+2\sigma R_\infty^2)t\leq\frac{\lambda}{2}\int_0^t D_s^2ds+\frac{h}{\lambda}-(2\kappa R_\infty^3+2\sigma R_\infty^4)t\\
&\leq\frac{h}{\lambda}+(4\sigma\lambda-2\kappa R_\infty^2-2\sigma R_\infty^2)t.
\end{aligned}$$

Therefore, for every $\lambda,h>0$ satisfying

$$\lambda<\frac{\kappa+\sigma}{2\sigma}R_\infty^2,\quad\frac{h}{\lambda}\leq\log\frac{1-\cos 2L_\infty}{1-\cos 2L_0}<\log\frac{1-\cos 2L_\infty}{1-\cos D(\Theta_{in})},\tag{4.3.8}$$

we have $\tau_{\lambda,h}\leq\tau$. In particular, if $\tau_{\lambda,h}=\infty$ for λ,h satisfying (4.3.8), we have

$$\sup_{0\leq t<\infty}D(\Theta_t)<2L_\infty\quad\text{and}\quad\limsup_{t\to\infty}D(\Theta_t)=0.$$

Finally, since

$$\mathbb{P}\left\{\sup_{0\leq t<\infty}D(\Theta_t)<2L_\infty\quad\text{and}\quad\limsup_{t\to\infty}D(\Theta_t)=0\right\}\geq 1-e^{-h}$$

for all λ,h satisfying (4.3.8), we conclude the desired probability estimate. □

Remark 4.3.1. *As $\frac{\sigma}{\kappa}\to 0$, the above probability lower bound*

$$1-\left(\frac{\sin^2 L_0}{\sin^2 L_\infty}\right)^{\frac{\kappa+\sigma}{2\sigma}R_\infty^2}$$

converges to 1. *This is also compatible with the asymptotic stability result of complete phase synchronization for the deterministic Kuramoto model (see Appendix B.2).*

4.4 Numerical simulations

In the sequel, we provide numerical examples for two-body and many-body systems and compare them with our analytical results. The following simulations are set in the domain of $[0, L_x] \times [0, L_y] = [0, 10] \times [0, 10]$, together with periodic boundary conditions. The coupling strength is fixed as $\kappa = 10$, and we vary the values of σ. The Milstein method is again implemented through all the numerical experiments with time-step $\Delta t = 0.001$.

4.4.1 A two-body system

In the bounded domain, the maximal distance between any two particles is $r_M = \sqrt{L_x^2 + L_y^2} = 10\sqrt{2}$. We consider the trigonometric communication weight function ψ:

$$\psi(r) = \frac{1}{\cos\left[\left(1 - \frac{r}{r_M}\right)\left(10^{-4} - \frac{\pi}{2}\right)\right]}. \tag{4.4.1}$$

Note that $\psi_{\min} = 1$, $\psi_{\max} = 10^4$, and

$$\int_0^{r_M} \psi^2(r)\,dr = \frac{r_M \tan\left(\frac{\pi}{2} - 10^{-4}\right)}{\frac{\pi}{2} - 10^{-4}} \approx 9.0037 \times 10^4.$$

Consider two-body system (4.2.1) with three different values of σ and $0 \le n \le 10$. For each σ, two particles are distributed in the domain randomly, and the initial relative heading angle $\theta_{in} \in \left(\frac{n\pi}{2}, \frac{(n+0.1)\pi}{2}\right)$ for each $n \in \mathbb{Z}$. We collect 100 sample simulations with the same time horizon $t = 10$. The mean and standard derivation are plotted in the following figures, together with the prediction θ_t given by Theorem 4.2.1 labeled with red circles.
Fig. 4.1 shows the numerical tests with different values of σ. Note that when $\kappa\psi_{\min} \le \sigma$, the mean values of θ_t is slightly different to the equilibrium in Theorem 4.2.1, and the standard derivation is larger.

Next, we test with the two-body system (4.2.5). One can observe that in the first figure of Fig. 4.2, the mean values of θ_t show a consistent asymptotic behavior with the theoretical prediction in Theorem 4.2.2 with a smaller

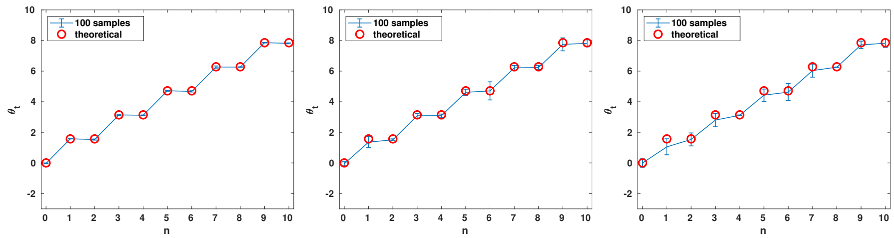


Figure 4.1: The initial angle $\theta_{in} = \frac{(n+0.1)\pi}{2}, \quad \kappa\psi_{\min} = 10.$

value of σ, even if it is far from $\kappa > \sigma\psi_{\max}$. This is because $\psi(\|x_t\|)$ cannot converge to zero but has a positive lower bound in our numeric simulation. In the second and third figure in Fig. 4.2, we increase σ even more, so that the discrepancy between the numerical tests and the equilibrium becomes significant, and the variance increasces with respect to the noise.

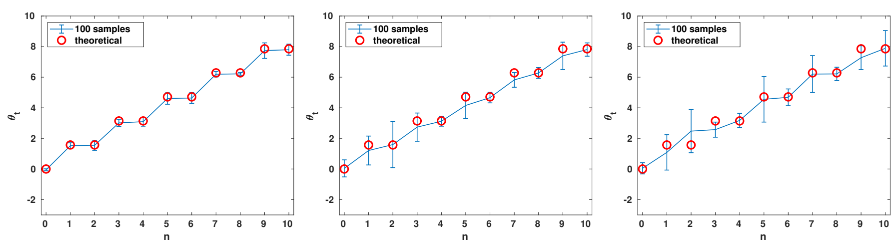


Figure 4.2: Noise depends on $\psi, \quad \kappa = 10.$

4.4.2 A many-body system

We consider a system (4.3.5) of $N = 100$ particles with two types of prepared initial data that lead to alignment and bipolar alignment, respectively. Note that the communication weight function is $\psi(r) \equiv 1$ here. The simulation results are taken from the average of 20 samples up to the final time $t = 4$.

As Fig. 4.3 shows, the heading angle alignment of Θ_t is formulated and stable for time $t \in [0, 4]$. The order parameter $R_2(\Theta_t) = R(2\Theta_t)$ approaches 1, and P_t approaches zero as we proved in Theorem 4.3.2. Also, they never touch to their limit in finite time. Note that although Theorem 4.3.2 only shows the convergence of $R_2(\Theta_t)$ to 1, $R(\Theta_t)$ also approached near 1 in our numerical simulations. This is because the stochastic stability of the heading angle alignment $\{\theta_t^j\}_{j=1}^N$, i.e., stability of $R(\Theta_t) = 1$ provided in Theorem 4.3.3. One can also observe that the average $R_2(\Theta_t)$ is monotonically increasing for $\kappa > \sigma$, which shows the sharpness of our result. Moreover, as Theorem 4.3.3 indicated, the third figure in Fig. 4.3 shows that the heading angle alignment of $\{\theta_t^j\}_{j=1}^N$ might occurs even when $\kappa \leq \sigma$.

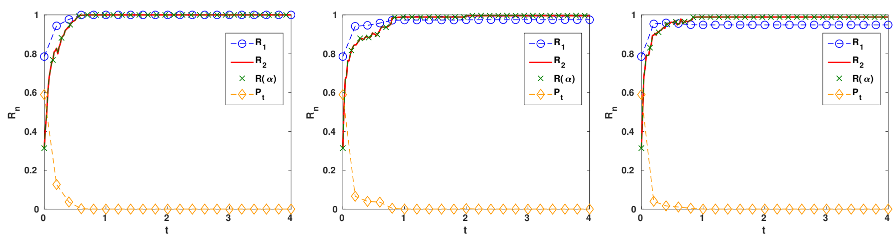


Figure 4.3: Alignment for $\kappa = 5$ and $\sigma = 1, 4, 6$.

Chapter 5

Conclusion and future works

In this thesis, we studied an emergent behavior of Inertial Spin model and two stochastic variants of Justh-Krishnaprasad model.

First, we studied the long-time behavior of deterministic IS model with nonzero damping. By adding nonzero damping, the IS model becomes dissipative from the original conservative model. As a consequence, if the communication weights are assumed to be positive constants with multiplicative structure $p_{ij} = p_i p_j$, their velocities and spins are always converges to an equilibrium regardless of initial data. Even more, we were able to prove that any equilibrium having non-parallel asymptotic velocities are linearly unstable.

Second, we implemented the sample path analysis for the stochastic persistency of the J-K model with additive white noise describing the emergence of heading angle alignment of J-K particles. Since the distribution of noise is uniform with respect to the angle configuration, there is a nontrivial possibility to stay in certain subset of phase space, where the drift effects are large enough to overcome the noise effect of each particles. For the additive noise J-K model, we also provided a possible upper bound of the expectation of order parameter squares. In this case, we were able to measure quantitatively the degree of the instability of alignment state induced by the additive noises.

Third, we presented the sufficient framework to obtain the asymptotic alignment of heading angles for the stochastic J-K model with multiplicative noise. Since the size of noise decreases near the alignment state, we were able to acheive the convegergence of system even in the presence of noise. We analyzed the evolution of expectation of $\log(R_t)$ to obtain the integrability of its drift terms, and finally conclude the almost surely convergence of heading angles to bipolar configurations. When the noises are imposed for the parameter κ, we also estimated the probability to converge to the heading angle alignment state.

However, there are some remaining issues to be explored further. For IS model, it is still not clear if all equilibria except flocking state $v_1 = \cdots = v_N$ are unstable. In our numerical simulations, we could not find any case showing the bipolar alignment of velocities, although we could not exclude them in our analysis. The asymptotic behavior of IS model with white noise [2] would be also interesting to consider. For additive noise J-K model, we may wonder what we can conclude as a corresponding result of sample path analysis for the instability of alignment state. Conversely, the stochastic persistency of the additive noise model mildly indicates the possibility to find an asymptotic lower bound on the expectation of order parameter square, so that the order parameter is oscillating in a certain region. For the multiplicative noise J-K models, we may consider the Vlasov-McKean limit to derive a corresponding mean field limit PDE and analyze the evolution of the expected order parameter. Similar problem can be also proposed for the additive noise J-K model, and it might be interesting to find an analogous upper bound of the expected order paremeter square for the additvie noise J-K PDE system. These topics will be studied in future works.

Appendix A

Theoretical backgrounds

A.1 Barbalat's lemma

Recall that any real valued sequence having convergent series converges to zero, i.e.,

$$\lim_{n\to\infty}\sum_{k=1}^{n} a_k \quad \text{exists} \quad \Rightarrow \quad \lim_{n\to\infty} a_n = 0.$$

The Barbalat's lemma gives an continuous analogue of this property, provided that the integrand is uniformly continuous.

Lemma A.1.1 ([5]). *Suppose that a real-valued function $f : [0,\infty) \to \mathbb{R}$ is uniformly continuous and satisfies*

$$\lim_{t\to\infty}\int_0^t f(s)ds \quad \textit{exists.}$$

Then, f tends to zero as $t \to \infty$:

$$\lim_{t\to\infty} f(t) = 0.$$

In fact, this uniform continuity condition cannot be replaced to a local regularity assumption. To see this, consider a following real-analytic function h:

$$h(x) = \sum_{k=1}^{\infty} e^{-(k^2x-k^3)^2}, \quad x \in \mathbb{R}.$$

Then, the function h above does not converges to zero, while the primitive of h is bounded and monotonically increasing.

On the other hand, the LaSalle's theorem does not require the uniform continuity, but some other extra conditions are needed.

Lemma A.1.2 ([63]). *Let $D \subset \mathbb{R}^d$ be a domain and $\Omega \subset D$ be a compact positively invariant set with respect to given equation*

$$\dot{x} = F(x). \tag{A.1.1}$$

Let $V : D \to \mathbb{R}$ be a C^1-function such that $\dot{V}(x) \leq 0$ in Ω, and E be the set of all points in Ω satisfying $\dot{V}(x) = 0$. Moreover, let M be the largest invariant set of (A.1.1) *in E. Then, every solution of* (A.1.1) *with initial data $x(0) \in \Omega$ approaches M as $t \to \infty$.*

A.2 Comparison principles for stochastic differential equations

In Chapter 3 and 4, we have frequently used the comparison principle of stochastic differential equations to obtain the desired results. The concrete statement of the most standard comparison principle is as follows.

Proposition A.2.1 ([60], Proposition 2.18 in Ch. 5). *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}$, and assume that we have a standard one-dimensional Brownian motion $\{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ and two continuous adapted process $X^{(j)}$; $j = 1, 2$, such that*

$$X_t^{(j)} = X_0^{(j)} + \int_0^t b_j(s, X_s^{(j)})ds + \int_0^t \sigma(s, X_s^{(j)})dW_s; \quad 0 \leq t < \infty$$

holds almost surely for $j = 1, 2$. Assume further that

1. *σ and b_j are continuous real-valued functions.*

2. *The dispersion matrix σ is x-Lipschitz.*

3. $X_0^{(1)} \le X_0^{(2)}$ *almost surely.*

4. $b_1 \le b_2$.

5. *Either b_1 or b_2 is x-Lipschitz .*

Then, we have

$$\mathbb{P}\left[X_t^{(1)} \le X_t^{(2)}, \quad \forall\ 0 \le t < \infty\right] = 1.$$

In our cases, initial data were always fixed as same, but there were certain difference on drift terms. Therefore, we can use this comparison principle without any modifications.

A.3 Well-posedness for stochastic differential equations

Similar to the deterministic ODE, there is an analogous well-posedness result of stochastic differential equations, provided that the drift and diffusion terms are sufficiently regular and small.

Lemma A.3.1 ([76]). *Consider a stochastic differential equation*

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s; \quad 0 \le t < \infty, \tag{A.3.1}$$

where b and σ are globally x-Lipschitz and there exists a positive constant M satisfying

$$\|b(t,x)\|_2^2 + \|\sigma(t,x)\|_2^2 \le M(1 + \|x\|_2^2)$$

for all $x \in \mathbb{R}^d$. Then, we have

1. *For any measurable initial condition X_0, there exists a unique solution X_t of* (A.3.1).

2. *We have*

$$\mathbb{E}\left[\sup_{t\le T} |X_t - Y_t|^2\right] \le 3e^{3L^2(T+4)T}\mathbb{E}|X_0 - Y_0|^2,$$

where L is the global Lipschitz constant of b, σ.

Therefore, since our SDEs (1.0.4)-(1.0.6) have globally Lipschitz continuous and uniformly bounded drift and diffusion, there exists a unique strong solution solving the corresponding SDEs.

A.4 Strong Markov Property

One might wonder if a conditional probability distribution of future process depends on its history or not. We say a stochastic process has a Markov property if the conditional distribution of future process only depends on the present time for every fixed reference time. Moreover, we say a process has a strong Markov property if the future distribution only depends on the present even for stopping times (optional times). The formal definitions of the stopping time, optional time and strong Markov Process are as follows.

Definition A.4.1. *Let $(\Omega, \mathcal{F})$ be a measurable space equipped with a filtration $\{\mathcal{F}_t\}$. A random variable $T : \Omega \to [0, \infty]$ is a stopping time of the filtration $\{\mathcal{F}_t\}$ if*

$$\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t, \quad \forall\ t \geq 0,$$

and the random variable T is an optional time of the filtration $\{\mathcal{F}_t\}$ if

$$\{\omega \in \Omega : T(\omega) < t\} \in \mathcal{F}_t, \quad \forall\ t \geq 0.$$

Note that the concepts of stopping time and the optional time are equivalent if $\{\mathcal{F}_t\}_{t\in I}$ is right-continuous, i.e.,

$$\bigcap_{t<s} \mathcal{F}_s = \mathcal{F}_t, \quad \forall\ t \in I.$$

Definition A.4.2. *Let μ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. A progressively measurable d-dimensional process*

$$X = \{X_t, \mathcal{F}_t; t \geq 0\}$$

on a probability space $(\Omega, \mathcal{F}, \mathbb{P}^\mu)$ is said to be a strong Markov process with initial distribution μ if

1. $\mathbb{P}^\mu[X_0 \in A] = \mu(A), \quad \forall\ A \in \mathcal{B}(\mathbb{R}^d).$

2. *For any optional time τ of $\{\mathcal{F}_t\}_{t\geq 0}$ and $A \in \mathcal{B}(\mathbb{R}^d)$,*

$$\mathbb{P}^\mu[X_{\tau+t} \in A|\mathcal{F}_{\tau+}] = \mathbb{P}^\mu[X_{\tau+t} \in A|X_\tau], \quad \mathbb{P}^\mu\text{-a.s. on } \{\tau < \infty\}.$$

A well-known result relating the Strong Markov Property and the solution of SDE is as below.

Lemma A.4.1 ([60])**.** *If the time-homogeneous SDE*

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s$$

admits a strong solution for every initial condition $x \in \mathbb{R}^d$, and if b, σ are locally bounded, then X_s has a strong Markov property.

Therefore, in our case, all solutions of (1.0.4)–(1.0.6) has a strong Markov property, and this justify the stopping time argument in Lemma 3.3.2.

Appendix B

The Kuramoto model

B.1 Basic descriptions

In Chapter 4, we presented some asymptotic stability results on J-K model with two types of multiplicative noises (1.0.8)–(1.0.9). For many-body systems, however, we mainly considered a specific case when ψ is defined as a constant function. Then, the system was reduced to a simpler form, namely the independent white noise model

$$\begin{cases} d\theta_t^j = \left(\frac{1}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^j)\right)\left(\kappa dt + \sqrt{2\sigma} dB_t^j\right), \\ \theta_0^j = \theta_{in}^j, \quad j = 1, \cdots, N, \end{cases} \tag{B.1.1}$$

and the dependent white noise model

$$\begin{cases} d\theta_t^j = \left(\frac{1}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^j)\right)\left(\kappa dt + \sqrt{2\sigma} dB_t\right), \\ \theta_0^j = \theta_{in}^j, \quad j = 1, \cdots, N. \end{cases} \tag{B.1.2}$$

Here, the drift part of both systems are already well-known as a Kuramoto model, which was first proposed from Yoshiki Kuramoto's work [61] in 1975.

The Kuramoto's derivation starts from the linearly coupled Stuart-Landau oscillators in [73], where each oscillator follows the following dynamics in the absence of coupling:

$$\dot{z} = (1 - |z|^2 + \sqrt{-1}\Omega)z, \quad z \in \mathbb{C}$$

Here, $\Omega \in \mathbb{R}$ is the natural frequency of the Stuart-Landau oscillator, which corresponds to the angular velocity of z in a polar coordinate form: for polar coordinate expression $z = re^{i\theta}$, the modulus r and the phase θ satisfies

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = \Omega.$$

Therefore, each Stuart-Landau oscillator has an unstable equilibrium $r = 0$ and a stable limit cycle $r = 1$, without any adjustment of the angular velocity (=natural frequency) Ω.

Then, we impose an all-to-all linear coupling to the system of N Stuart-Landau oscillators

$$\frac{dz_j}{dt} = (1 - |z_j|^2 + \sqrt{-1}\Omega_j)z_j + \frac{\kappa}{N}\sum_{i=1}^{N}(z_i - z_j), \quad j = 1, \cdots, N,$$

and use the polar coordinate expression $z_j = r_j e^{\sqrt{-1}\theta_j}$ for each j to obtain

$$\dot{r}_j + \sqrt{-1}r_j\dot{\theta}_j = \left(1 - r_j^2 + \sqrt{-1}\Omega_j\right) r_j + \frac{\kappa}{N}\sum_{i=1}^{N}\left(r_i e^{\sqrt{-1}(\theta_i - \theta_j)} - r_j\right),$$

or equivalently,

$$\begin{cases} \dot{r}_j = (1 - r_j^2)r_j + \dfrac{\kappa}{N}\displaystyle\sum_{i=1}^{N}\left(r_i\cos(\theta_i - \theta_j) - r_j\right), \quad j = 1, \cdots, N, \\ \dot{\theta}_j = \Omega_j + \dfrac{\kappa}{N}\displaystyle\sum_{i=1}^{N}\dfrac{r_i}{r_j}\sin(\theta_i - \theta_j), \quad j = 1, \cdots, N. \end{cases} \tag{B.1.3}$$

We then assume that all oscillators are confined in the stable limit cycle (i.e., $r_j \sim 1$), and denote the resulted equation as the Kuramoto model:

$$\dot{\theta}_j = \Omega_j + \frac{\kappa}{N}\sum_{i=1}^{N}\sin(\theta_i - \theta_j), \quad j = 1, \cdots, N. \tag{B.1.4}$$

Therefore, the drift part of (B.1.1) and (B.1.2) are exactly coincides with (B.1.4) when all natural frequencies $\Omega_1, \cdots, \Omega_N$ are identially zero.

B.2 Previous results

We now introduce some previous analytic results on the Kuramoto model (B.1.4) to compare with our results in Chapter 3 and Chapter 4. Since (B.1.1) and (B.1.2) are correspond to (B.1.3) only when $\Omega_1 = \cdots = \Omega_N = 0$, we only present previous results for identical natural frequencies in literature. To see the most recent result on the non-idnetical Kuramoto ensemble, see [55].

First, we begin with a dynamics of order parameter R.

Proposition B.2.1. *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution of* (B.1.4)*, where the system parameters $\Omega = (\Omega_1, \cdots, \Omega_N)$ and κ satisfy*

$$\Omega_1 = \cdots = \Omega_N = 0, \quad \kappa > 0.$$

Then, the order parameter R_t is monotonically increasing in t. Moreover, if initial order paramter

$$R_{in} := R(\Theta(0))$$

is strictly positive, then the order parameter $R_t := R(\Theta(t))$ is strictly increasing in t.

Lemma 4.3.2 shows that for multiplicative noise models (B.1.1) and (B.1.2), R_t is a submartingale process and $R_t > 0$ holds for every finite time t almost surely provided that $R_0 > 0$. Therefore, Proposition B.2.1 can be regarded as a special case $\sigma = 0$ of Lemma 4.3.2.

Now, recall that we provided a sufficient framework leading to the convergence of order parameter R to a positive random variable and the convergence of bipolar order parameter R_2 to 1 almost surely (Theorem 4.3.1 and Theorem 4.3.2). The corresponding result for deterministic Kuramoto model (B.1.4) says that there are only few bipolar states that can be reached from generic initial configurations.

Theorem B.2.1 ([9])**.** *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution of* (B.1.4)*, where the system parameters $\Omega = (\Omega_1, \cdots, \Omega_N)$ and κ satisfy*

$$\Omega_1 = \cdots = \Omega_N = 0, \quad \kappa > 0.$$

Suppose that the initial phase configuration $\Theta_0 := (\theta_1(0), \cdots, \theta_N(0))$ *satisfies*

$$R(\Theta_0) > 0, \quad \theta_i(0) \neq \theta_j(0) \quad \textit{modulo } 2\pi \quad \forall\ i \neq j.$$

Then, for each $i = 1, \cdots, N$, *the phase* θ_i *converges*

$$\theta_i^\infty := \lim_{t\to\infty} \theta_i(t),$$

and there exists $\phi^\infty \in \mathbb{R}$ *satisfying*

$$|\{i : \theta_i^\infty - \phi^\infty = 0 \mod 2\pi\}| \geq N-1, \quad |\{i : \theta_i^\infty - \phi^\infty = \pi \mod 2\pi\}| \leq 1.$$

Finally, in Theorem 4.3.3, we were able to find a lower bound of probability to achieve $D(\Theta(t)) \to 0$ (or $R_t \to 1$) for the model (B.1.2), provided that $D(\Theta_{in})$ is smaller than π. As we noted in Remark 4.3.1, the lower bound converges to 1 as $\frac{\sigma}{\kappa}$ tends to zero. Again, we have an analogous result for (B.1.4), which is consistent with Theorem 4.3.3.

Theorem B.2.2 ([42])**.** *Let* $\Theta = (\theta_1, \cdots, \theta_N)$ *be a solution of* (B.1.4), *where the system parameters* $\Omega = (\Omega_1, \cdots, \Omega_N)$ *and* κ *satisfy*

$$\Omega_1 = \cdots = \Omega_N = 0, \quad \kappa > 0.$$

Suppose that the initial phase configuration $\Theta_0 := (\theta_1(0), \cdots, \theta_N(0))$ *satisfies*

$$D(\Theta(0)) = \max_{i,j} |\theta_i(0) - \theta_j(0)| < \pi.$$

Then, $D(\Theta(t))$ *decreases monotonically and converges to zero as* $t \to \infty$.

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국문초록

본 학위 논문에서는, 플로킹 현상을 기술하는 관성 스핀 모형과 그로부터 유도된 확률 모형들에 대하여 연구한다. 먼저 우리는 관성 스핀 모형의 정당화를 위해 3차원 공간에서의 해밀토니언 역학적 관점에 따라 속력을 보존하는 플로킹 모형을 구현하는 방법을 소개하고 그 집단 행동을 분석한다. 그러나 관성 스핀 모형의 집단현상을 실제와 비교함에 있어 입자들의 다이나믹스에 영향을 줄 수 있는 미지의 불확실성을 고려하는 것이 보다 자연스러우므로, 우리는 자연계의 창발 현상을 보다 정확히 설명하기 위해 관성 스핀 모형에 백색소음을 추가하는 여러 가지 방법을 고려한다. 우리가 고려하는 확률 모형은 크게 두 가지인데, 2차원 관성 스핀 모형의 극소 관성 체제 하에서 자연스럽게 유도되는 J-K 모형에 백색 소음을 직접 추가하는 가법적 방법과, J-K 모형의 상호 작용 계수에 백색 소음을 추가하는 곱셈적 방법으로 나뉘어 진다. 가법적 백색소음이 있는 J-K 모형에서는, 주어진 시간 동안 각각의 표본 경로가 플로킹 상태에서 일정 이상 멀어질 수 있는 확률을 어림하고, 또 플로킹 상태에서 가까워짐을 나타내는 질서도의 점근적 상극한을 계산하여 결정론적 J-K 모형과의 차이점을 나타낸다. 그와 반대로 곱셈적 백색소음이 있는 J-K 모형에서는 결정론적 J-K 모형과 같이 플로킹 상태의 점근적 안정성이 발생할 수 있는 충분조건에 대해 공부한다.

주요어휘: 플로킹, 관성 스핀 모델, 임의적 동역학계, 확률미분방정식
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